Computational Manifolds and Applications - 2011

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Problem 1

Consider the parametric surface given by

$$
\begin{aligned}
x(u,v) &= \frac{4v(u^2+v^2-1)}{(u^2+v^2+1)^2},\\
y(u,v) &= \frac{4u(u^2+v^2-1)}{(u^2+v^2+1)^2},\\
z(u,v) &= \frac{4(u^2-v^2)}{(u^2+v^2+1)^2}.
\end{aligned}
$$

(a) Let's make the change of variables

$$
\begin{aligned}
x &= \rho\cos\theta\\
y &= \rho\sin\theta.
\end{aligned}
$$

For x we have:

$$
\begin{aligned}
x(\rho,\theta) &= \frac{4(\rho\sin\theta)((\rho\cos\theta)^2+(\rho\sin\theta)^2-1)}{((\rho\cos\theta)^2+(\rho\sin\theta)^2+1)^2},\\
&= \frac{4\rho(\rho^2\cos^2\theta+\rho^2\sin^2\theta-1)}{(\rho^2\cos^2\theta+\rho^2\sin^2\theta+1)^2}\sin\theta,\\
&= \frac{4\rho(\rho^2(\cos^2\theta+\sin^2\theta)-1)}{(\rho^2(\cos^2\theta+\sin^2\theta)+1)^2}\sin\theta,\\
&= \frac{4\rho(\rho^2-1)}{(\rho^2+1)^2}\sin\theta.
\end{aligned}
$$

For y we have:

$$
\begin{aligned}
y(\rho,\theta) &= \frac{4(\rho\cos\theta)((\rho\cos\theta)^2+(\rho\sin\theta)^2-1)}{((\rho\cos\theta)^2+(\rho\sin\theta)^2+1)^2},\\
&= \frac{4\rho(\rho^2\cos^2\theta+\rho^2\sin^2\theta-1)}{(\rho^2\cos^2\theta+\rho^2\sin^2\theta+1)^2}\cos\theta,\\
&= \frac{4\rho(\rho^2-1)}{(\rho^2+1)^2}\cos\theta.
\end{aligned}
$$

And for z:

$$\begin{aligned}
z(\rho,\theta) &= \frac{4((\rho\cos\theta)^2 - (\rho\sin\theta)^2)}{((\rho\cos\theta)^2 + (\rho\sin\theta)^2 + 1)^2} \\
&= \frac{4((\rho\cos\theta)^2 - (\rho\sin\theta)^2)}{(\rho^2+1)^2} \\
&= \frac{4\rho^2(\cos^2\theta - \sin^2\theta)}{(\rho^2+1)^2} \\
&= \frac{4\rho^2}{(\rho^2+1)^2}\cos 2\theta.
\end{aligned}$$

We can define

$$r(\rho) = \frac{4\rho(\rho^2-1)}{(\rho^2+1)^2},$$

so we have $x(\rho,\theta) = r(\rho)\sin\theta$ and $y(\rho,\theta) = r(\rho)\cos\theta$. Observe that, for $\rho \neq 0$:

$$\begin{aligned}
r(1/\rho) &= \frac{4(1/\rho)((1/\rho)^2 - 1)}{((1/\rho)^2+1)^2} \\
&= \frac{4(1/\rho)(1/\rho^2)(1-\rho^2)}{((1/\rho)^2(1+\rho^2))^2} \\
&= \frac{4(1/\rho^3)(1-\rho^2)}{(1/\rho^4)(1+\rho^2)^2} \\
&= \frac{4\rho(1-\rho^2)}{(1+\rho^2)^2} \\
&= -r(\rho),
\end{aligned}$$

so we have

$$\begin{aligned}
x(1/\rho,\theta) &= r(1/\rho)\sin\theta = -r(\rho)\sin\theta = -x(\rho,\theta), \\
y(1/\rho,\theta) &= r(1/\rho)\cos\theta = -r(\rho)\cos\theta = -y(\rho,\theta).
\end{aligned}$$

Also we have

$$\begin{aligned}
z(1/\rho,\theta) &= \frac{4(1/\rho)^2}{((1/\rho)^2+1)^2}\cos 2\theta \\
&= \frac{4(1/\rho^2)}{(1/\rho^4)(1+\rho^2)^2}\cos 2\theta \\
&= \frac{4\rho^2}{(1+\rho^2)^2}\cos 2\theta \\
&= z(\rho,\theta).
\end{aligned}$$

For any θ find an integer k such that the the angle $\theta_0 = \theta + \pi + 2k\pi$ belongs to $[-\pi,\pi]$, then we have

$$\begin{aligned}
x(\rho,\theta_0) &= r(\rho)\sin\theta_0 \\
&= r(\rho)\sin(\theta+\pi+2k\pi) \\
&= r(\rho)\sin(\theta+\pi) \\
&= -r(\rho)\sin\theta \\
&= -x(\rho,\theta).
\end{aligned}$$

$$
\begin{aligned}
y(\rho, \theta_0) &= r(\rho) \cos \theta_0 \\
&= r(\rho) \cos(\theta + \pi + 2k\pi) \\
&= r(\rho) \cos(\theta + \pi) \\
&= -r(\rho) \cos \theta \\
&= -y(\rho, \theta).
\end{aligned}
$$

$$
\begin{aligned}
z(\rho, \theta_0) &= \frac{4\rho^2}{(\rho^2+1)^2} \cos 2\theta_0 \\
&= \frac{4\rho^2}{(\rho^2+1)^2} \cos 2(\theta + \pi + 2k\pi) \\
&= \frac{4\rho^2}{(\rho^2+1)^2} \cos(2\theta + 2\pi + 4k\pi) \\
&= \frac{4\rho^2}{(\rho^2+1)^2} \cos(2\theta) \\
&= z(\rho, \theta).
\end{aligned}
$$

Then for a given $\rho > 1$ and any θ, we can define $\rho_0 = 1/\rho$ and $\theta_0 = \theta + \pi + 2k\pi$, so we have $\rho_0 \in [0, 1]$, $\theta_0 \in [-\pi, \pi]$ and

$$x(\rho_0, \theta_0) = -x(\rho_0, \theta) = -x(1/\rho, \theta) = x(\rho, \theta)$$

$$y(\rho_0, \theta_0) = -y(\rho_0, \theta) = -y(1/\rho, \theta) = y(\rho, \theta)$$

$$z(\rho_0, \theta_0) = z(\rho_0, \theta) = z(1/\rho, \theta) = z(\rho, \theta).$$

Then the entire trace of the surface can be obtained for $\rho \in [0, 1]$, $\theta \in [-\pi, \pi]$.

A plot of this surface using $\rho \in [0, 1]$ and $\theta \in [-\pi, \pi]$ can be seen in Figure 1.

A plot of this surface using $\rho \in [0, 1]$ and $\theta \in [0, \pi]$ can be seen in Figure 2.

Let's calculate the singular points of this surface. The tangent vector in the direction of ρ is given by

$$
\begin{aligned}
x_\rho(\rho, \theta) &= r'(\rho) \sin \theta \\
y_\rho(\rho, \theta) &= r'(\rho) \cos \theta \\
z_\rho(\rho, \theta) &= s'(\rho) \cos 2\theta,
\end{aligned}
$$

where $s(\rho) = \frac{4\rho^2}{(\rho^2+1)^2}$. This tangent vector will be zero when $r'(\rho) = 0$, and $s'(\rho) \cos 2\theta = 0$.

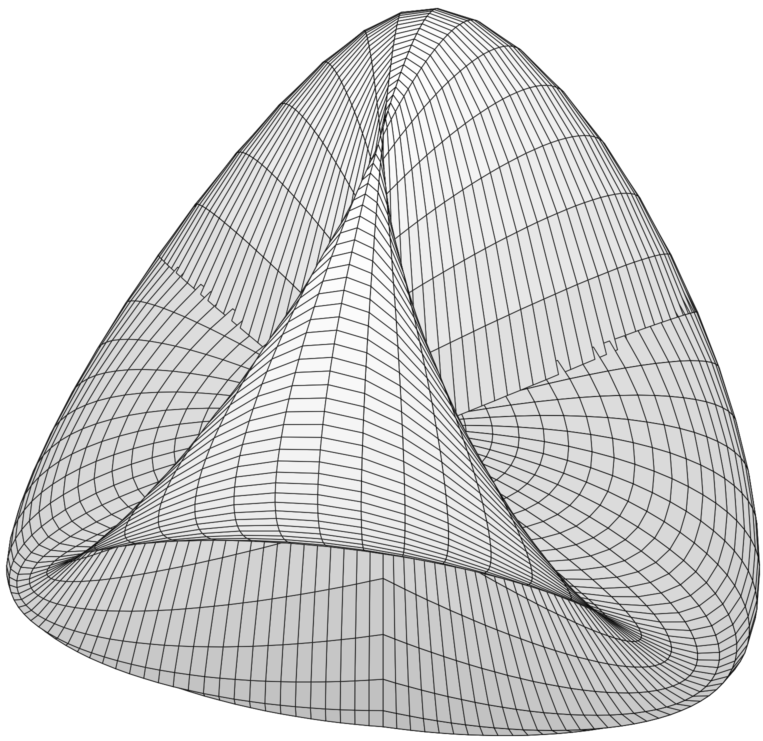


Figure 1: Plot of the surface.

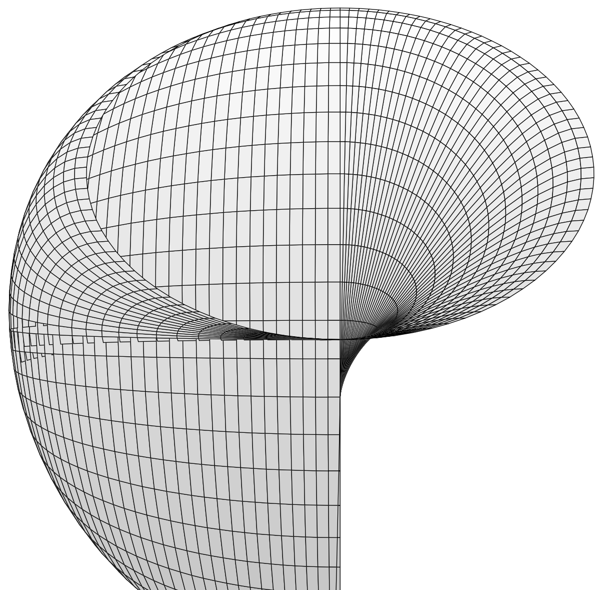


Figure 2: Plot of the surface using $\rho \in [0, 1]$ and $\theta \in [0, \pi]$.

So we have

$$\begin{aligned}
0 &= r'(\rho) \\
&= \frac{d}{d\rho}\left(\frac{4\rho(\rho^2-1)}{(\rho^2+1)^2}\right) = 4\frac{d}{d\rho}\left(\frac{\rho^3-\rho}{(\rho^2+1)^2}\right) \\
&= 4\frac{\left(\frac{d}{d\rho}(\rho^3-\rho)\right)(\rho^2+1)^2 - (\rho^3-\rho)\left(\frac{d}{d\rho}(\rho^2+1)^2\right)}{(\rho^2+1)^4} \\
&= 4\frac{(3\rho^2-1)(\rho^2+1)^2 - (\rho^3-\rho)(2(\rho^2+1)\cdot 2\rho)}{(\rho^2+1)^4} \\
&= 4\frac{(3\rho^2-1)(\rho^2+1)^2 - (\rho^3-\rho)(2(\rho^2+1)\cdot 2\rho)}{(\rho^2+1)^4} \\
&= 4\frac{(3\rho^2-1)(\rho^2+1) - 4(\rho^4-\rho^2)}{(\rho^2+1)^3} \\
&= 4\frac{3\rho^4-\rho^2+3\rho^2-1-4\rho^4+4\rho^2)}{(\rho^2+1)^3} \\
&= 4\frac{-\rho^4+6\rho^2-1}{(\rho^2+1)^3}
\end{aligned}$$

then we have

$$\begin{aligned}
-\rho^4+6\rho^2-1 &= 0 \\
4\rho^2 &= \rho^4-2\rho^2+1 \\
4\rho^2 &= (\rho^2-1)^2 \\
2\rho &= |\rho^2-1|,
\end{aligned}$$

then $\rho^2-2\rho-1=0$ or $\rho^2+2\rho-1$, solving the first equation we obtain:

$$\rho = \frac{2\pm\sqrt{(-2)^2-4(-1)}}{2} = \frac{2\pm\sqrt{8}}{2} = \frac{2\pm 2\sqrt{2}}{2} = 1\pm\sqrt{2},$$

and for the second equation:

$$\rho = \frac{-2\pm\sqrt{2^2-4(-1)}}{2} = \frac{-2\pm\sqrt{8}}{2} = \frac{-2\pm 2\sqrt{2}}{2} = -1\pm\sqrt{2},$$

So we have four possibilities: $\rho_0 = -1+\sqrt{2} \approx 0.4142$, $\rho_1 = 1+\sqrt{2} \approx 2.4142$. $\rho_2 = -1-\sqrt{2} \approx -2.4142$ and $\rho_3 = 1-\sqrt{2} \approx -0.4142$. But we only need to consider $\rho \in [0,1]$, so the only viable solution is ρ_0.

Observe now that

$$\begin{aligned}
s'(\rho) &= \frac{d}{d\rho}\left(\frac{4\rho^2}{(\rho^2+1)^2}\right) \\
&= 4\frac{2\rho(\rho^2+1)^2 - \rho^2\cdot 2(\rho^2+1)\cdot 2\rho}{(\rho^2+1)^4} \\
&= 4\frac{2\rho(\rho^2+1)^2 - 4\rho^4(\rho^2+1)}{(\rho^2+1)^4} \\
&= 4\frac{2\rho(\rho^2+1) - 4\rho^4}{(\rho^2+1)^3}.
\end{aligned}$$

But $s'(\rho_0) \neq 0$, then to obtain $s'(\rho_0)\cos 2\theta = 0$, we must have $\cos 2\theta = 0$, which (for $\theta \in [-\pi, \pi]$) is satisfied for $\theta_0 = \pi/4$ and $\theta_1 = -\pi/4$.

Then we have these two singular points: $(\rho_0, \theta_0), (\rho_0, \theta_1)$.

Now let's check the derivatives in relation to θ:

$$\begin{aligned} x_\theta(\rho,\theta) &= r(\rho)\cos\theta \\ y_\theta(\rho,\theta) &= -r(\rho)\sin\theta \\ z_\theta(\rho,\theta) &= -2s(\rho)\sin 2\theta, \end{aligned}$$

the zero vector will happen for $r(\rho) = 0$ and $s(\rho)\sin 2\theta = 0$, then

$$\begin{aligned} r(\rho) &= 0 \\ \frac{4\rho(\rho^2-1)}{(\rho^2+1)^2} &= 0, \end{aligned}$$

so $\rho = \rho_5 = 0$ or $\rho = \rho_6 = 1$.

When $\rho = \rho_5 = 0$, we are at the point $u = 0, v = 0$.

For $\rho = \rho_6 = 1$, we have $s(\rho) = 1 \neq 0$, then we must have $\sin 2\theta = 0$, which happens for $\theta \in \{0, -\pi/2, \pi/2\}$, but observe that

$$\begin{aligned} x(1,\pi/2) &= x(1,-\pi/2) \\ y(1,\pi/2) &= y(1,-\pi/2) \\ z(1,\pi/2) &= z(1,-\pi/2) \end{aligned}$$

so these two points are actually a single point of self-intersection.

Then, there are five singular points:

$$\begin{gathered} \left(x\left(-1+\sqrt{2},-\pi/4\right), y\left(-1+\sqrt{2},-\pi/4\right), z\left(-1+\sqrt{2},-\pi/4\right)\right) \\ \left(x\left(-1+\sqrt{2},\pi/4\right), y\left(-1+\sqrt{2},\pi/4\right), z\left(-1+\sqrt{2},\pi/4\right)\right) \\ (x(0,0), y(0,0), z(0,0)) \\ (x(1,0), y(1,0), z(1,0)) \\ (x(1,\pi/2), y(1,\pi/2), z(1,\pi/2)). \end{gathered}$$

(b) Now let's make $u = \tan(\theta/2)$ and $v = \rho$. Observe that:

$$\tan\theta = \tan\left(2\frac{\theta}{2}\right) = \frac{2\tan(\theta/2)}{1-\tan^2(\theta/2)} = \frac{2u}{1-u^2},$$

and

$$\sin\theta = \sin\left(2\frac{\theta}{2}\right) = \frac{2\tan(\theta/2)}{1+\tan^2(\theta/2)} = \frac{2u}{1+u^2} = \frac{2u(1+u^2)}{(1+u^2)^2},$$

then we have

$$\begin{aligned} x(u,v) &= \frac{4v(v^2-1)}{(v^2+1)^2}\sin\theta \\ &= \frac{4v(v^2-1)}{(v^2+1)^2}\frac{2u(1+u^2)}{(1+u^2)^2} \\ &= \frac{8uv(1+u^2)(v^2-1)}{(1+u^2)^2(v^2+1)^2}. \end{aligned}$$

Observe now that

$$\begin{aligned} \cos\theta &= \sin\theta/\tan\theta \\ &= \frac{2u}{1+u^2}\left(\frac{2u}{1-u^2}\right)^{-1} \\ &= \left(\frac{2u}{1+u^2}\right)\left(\frac{1-u^2}{2u}\right) \\ &= \frac{1-u^2}{1+u^2} \\ &= \left(\frac{1-u^2}{1+u^2}\right)\left(\frac{1+u^2}{1+u^2}\right) \\ &= \frac{1-u^4}{(1+u^2)^2}. \end{aligned}$$

so

$$\begin{aligned} y(u,v) &= \frac{4v(v^2-1)}{(v^2+1)^2}\cos\theta \\ &= \left(\frac{4v(v^2-1)}{(v^2+1)^2}\right)\left(\frac{1-u^4}{(1+u^2)^2}\right) \\ &= \frac{4v(1-u^4)(v^2-1)}{(v^2+1)^2(1+u^2)^2} \end{aligned}$$

And we have

$$\begin{aligned} \cos 2\theta &= 1-2\sin^2\theta \\ &= 1-2\left(\frac{2u}{1+u^2}\right)^2 \\ &= 1-\frac{8u^2}{(1+u^2)^2} \\ &= \frac{(1+u^2)^2-8u^2}{(1+u^2)^2} \\ &= \frac{1+2u^2+u^4-8u^2}{(1+u^2)^2} \\ &= \frac{1-6u^2+u^4}{(1+u^2)^2}, \end{aligned}$$

then

$$
\begin{aligned}
z(\rho,\theta) &= \frac{4v^2}{(v^2+1)^2}\cos 2\theta \\
&= \frac{4v^2(1-6u^2+u^4)}{(1+u^2)^2(v^2+1)^2}
\end{aligned}
$$

Problem 2

Consider the parametric surface given by

$$
\begin{aligned}
x &= \frac{4u(1-u^2)((a+r)v^4+2(a-3r)v^2+a+r)}{(u^2+1)^2(v^2+1)^2}, \\
y &= \frac{4rv(1-u^4)(1-v^2)}{(u^2+1)^2(v^2+1)^2}, \\
z &= \frac{8ruv(1+u^2)(1-v^2)}{(u^2+1)^2(v^2+1)^2},
\end{aligned}
$$

where $a > r > 1$.

Observe that for $u \neq 0$:

$$
\begin{aligned}
x(1/u,v) &= \frac{4(1/u)(1-(1/u)^2)((a+r)v^4+2(a-3r)v^2+a+r)}{((1/u)^2+1)^2(v^2+1)^2} \\
&= \frac{4(1/u)(1-(1/u)^2)((a+r)v^4+2(a-3r)v^2+a+r)}{((1/u)^2(1+u^2))^2(v^2+1)^2} \\
&= \frac{4(1/u)(1-(1/u)^2)((a+r)v^4+2(a-3r)v^2+a+r)}{(1/u)^4(1+u^2)^2(v^2+1)^2} \\
&= u^4\left(\frac{4(1/u)(1-(1/u)^2)((a+r)v^4+2(a-3r)v^2+a+r)}{(1+u^2)^2(v^2+1)^2}\right) \\
&= \frac{4u(u^2-1)((a+r)v^4+2(a-3r)v^2+a+r)}{(1+u^2)^2(v^2+1)^2} \\
&= \frac{4(-u)(1-u^2)((a+r)v^4+2(a-3r)v^2+a+r)}{(1+u^2)^2(v^2+1)^2} \\
&= x(-u,v).
\end{aligned}
$$

For $u > 1$ we have $1/u \in [0,1]$, so $-1/u \in [-1,0] \subset [-1,1]$. For $u < -1$ we have $-1/u \in [0,1] \subset [-1,1]$. Then, for any u, we have $x(u,v) = x(u_0,v)$, where $u_0 \in [-1,1]$ is defined by

$$
u_0 = \begin{cases} u & \text{if } u \in [-1,1], \\ -1/u & \text{if } u \notin [-1,1]. \end{cases}
$$

Observe now that

$$\begin{aligned}
x(u,1/v) &= \frac{4u(1-u^2)((a+r)(1/v)^4+2(a-3r)(1/v)^2+a+r)}{(u^2+1)^2((1/v)^2+1)^2}, \\
&= \frac{4u(1-u^2)((a+r)(1/v)^4+2(a-3r)(1/v)^2+a+r)}{(u^2+1)^2(1/v^4)(v^2+1)^2}, \\
&= v^4\left(\frac{4u(1-u^2)((a+r)(1/v)^4+2(a-3r)(1/v)^2+a+r)}{(u^2+1)^2(v^2+1)^2}\right), \\
&= \frac{4u(1-u^2)((a+r)+2(a-3r)v^2+(a+r)v^4)}{(u^2+1)^2(v^2+1)^2}, \\
&= x(u,v).
\end{aligned}$$

Then, for any v, we have $x(u,v) = x(u,v_0)$, where $v_0 \in [-1,1]$ is defined by

$$v_0 = \begin{cases} v & \text{if } v \in [-1,1], \\ 1/v & \text{if } v \notin [-1,1]. \end{cases}$$

For y we have

$$\begin{aligned}
y(1/u,v) &= \frac{4rv(1-(1/u)^4)(1-v^2)}{((1/u)^2+1)^2(v^2+1)^2} \\
&= \frac{4rv(1-(1/u)^4)(1-v^2)}{(1/u)^4(1+u^2)^2(v^2+1)^2} \\
&= u^4\left(\frac{4rv(1-(1/u)^4)(1-v^2)}{(1+u^2)^2(v^2+1)^2}\right) \\
&= \frac{4rv(u^4-1)(1-v^2)}{(1+u^2)^2(v^2+1)^2} \\
&= \frac{4r(-v)(1-u^4)(1-v^2)}{(1+u^2)^2(v^2+1)^2} \\
&= y(u,-v).
\end{aligned}$$

Then, for any u, we have $y(u,v) = y(u_0,-v)$, where $u_0 \in [-1,1]$ is defined by

$$u_0 = \begin{cases} u & \text{if } u \in [-1,1], \\ 1/u & \text{if } u \notin [-1,1]. \end{cases}$$

Also,

$$\begin{aligned}
y(u,1/v) &= \frac{4r(1/v)(1-u^4)(1-(1/v)^2)}{(u^2+1)^2((1/v)^2+1)^2} \\
&= \frac{4r(1/v)(1-u^4)(1-(1/v)^2)}{(u^2+1)^2(1/v^4)(1+v^2)^2} \\
&= v^4\left(\frac{4r(1/v)(1-u^4)(1-(1/v)^2)}{(u^2+1)^2(1+v^2)^2}\right) \\
&= \frac{4rv(1-u^4)(v^2-1)}{(u^2+1)^2(1+v^2)^2} \\
&= \frac{4r(-v)(1-u^4)(1-v^2)}{(u^2+1)^2(1+v^2)^2} \\
&= y(u,-v).
\end{aligned}$$

Then, for any v, we have $y(u,v) = y(u,v_0)$, where $v_0 \in [-1,1]$ is defined by

$$v_0 = \begin{cases} v & \text{if } v \in [-1,1], \\ -1/v & \text{if } v \notin [-1,1]. \end{cases}$$

And finally,

$$\begin{aligned} z(1/u,v) &= \frac{8r(1/u)v(1+(1/u)^2)(1-v^2)}{((1/u)^2+1)^2(v^2+1)^2} \\ &= \frac{8r(1/u)v(1+(1/u)^2)(1-v^2)}{(1/u^4)(u^2+1)^2(v^2+1)^2} \\ &= u^4\left(\frac{8r(1/u)v(1+(1/u)^2)(1-v^2)}{(u^2+1)^2(v^2+1)^2}\right) \\ &= \frac{8ruv(u^2+1)(1-v^2)}{(u^2+1)^2(v^2+1)^2} \\ &= z(u,v). \end{aligned}$$

Then, for any u, we have $z(u,v) = z(u_0,v)$, where $u_0 \in [-1,1]$ is defined by

$$u_0 = \begin{cases} u & \text{if } u \in [-1,1], \\ 1/u & \text{if } u \notin [-1,1]. \end{cases}$$

And,

$$\begin{aligned} z(u,1/v) &= \frac{8ru(1/v)(1+u^2)(1-(1/v)^2)}{(u^2+1)^2((1/v)^2+1)^2} \\ &= \frac{8ru(1/v)(1+u^2)(1-(1/v)^2)}{(u^2+1)^2(1/v)^2(1+v^2)^2} \\ &= v^4\left(\frac{8ru(1/v)(1+u^2)(1-(1/v)^2)}{(u^2+1)^2(1/v)^2(1+v^2)^2}\right) \\ &= \frac{8ruv(1+u^2)(v^2-1)}{(u^2+1)^2(1/v)^2(1+v^2)^2} \\ &= \frac{8ru(-v)(1+u^2)(1-v^2)}{(u^2+1)^2(1/v)^2(1+v^2)^2} \\ &= z(u,-v). \end{aligned}$$

Then, for any v, we have $z(u,v) = z(u,v_0)$, where $v_0 \in [-1,1]$ is defined by

$$v_0 = \begin{cases} v & \text{if } v \in [-1,1], \\ -1/v & \text{if } v \notin [-1,1]. \end{cases}$$

With this we conclude that the entire surface is obtained for $u, v \in [-1,1]$

A plot of this surface using $a = 2$ and $r = 1$ can be seen in Figure 3.

It is easy to see that for $u \in [-1,1]$

$$\begin{aligned} x(u,1) &= x(u,-1) \\ y(u,1) &= 0 = y(u,-1) \\ z(u,1) &= 0 = z(u,-1), \end{aligned}$$

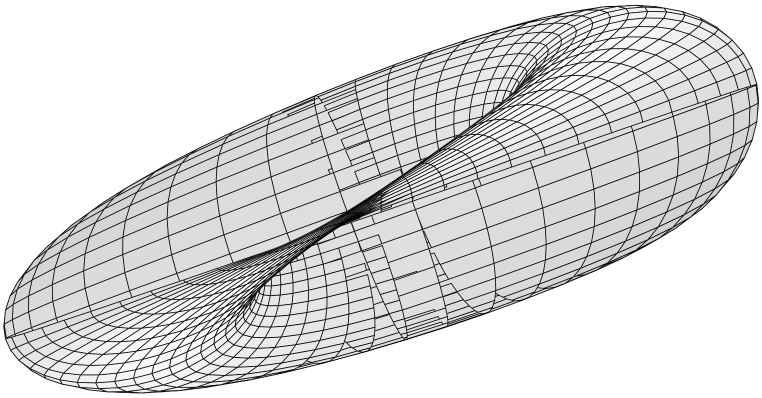


Figure 3: Plot of the surface.

so the points of this surface at $(u, 1)$ are glued at the points $(u, -1)$.

Notice also that for $v \in [-1, 1]$

$$\begin{aligned} x(1, v) &= 0 = x(-1, -v) \\ y(1, v) &= 0 = y(-1, -v) \\ z(1, v) &= z(-1, -v), \end{aligned}$$

then the points of this surface at $(1, v)$ are glued at the points $(-1, -v)$. Therefore we see that this surface is a Klein bottle (projected in $\mathbb{R}^3$).

Problem 3

Consider the parametric surface given by

$$\begin{aligned} x &= \frac{(u^4 - 6u^2 + 1)((a + r)v^4 + 2(a - 3r)v^2 + a + r)}{(u^2 + 1)^2(v^2 + 1)^2}, \\ y &= \frac{4u(1 - u^2)((a + r)v^4 + 2(a - 3r)v^2 + a + r)}{(u^2 + 1)^2(v^2 + 1)^2}, \\ z &= \frac{8ruv(1 + u^2)(1 - v^2)}{(u^2 + 1)^2(v^2 + 1)^2}, \end{aligned}$$

where $a > r > 1$.

Observe that for $u \neq 0$:

$$\begin{aligned} x(1/u, v) &= \frac{((1/u)^4 - 6(1/u)^2 + 1)((a + r)v^4 + 2(a - 3r)v^2 + a + r)}{((1/u)^2 + 1)^2(v^2 + 1)^2} \\ &= \frac{((1/u)^4 - 6(1/u)^2 + 1)((a + r)v^4 + 2(a - 3r)v^2 + a + r)}{(1/u)^4(1 + u^2)^2(v^2 + 1)^2} \\ &= u^4 \left(\frac{((1/u)^4 - 6(1/u)^2 + 1)((a + r)v^4 + 2(a - 3r)v^2 + a + r)}{(1 + u^2)^2(v^2 + 1)^2} \right) \\ &= \frac{(1 - 6u^2 + u^4)((a + r)v^4 + 2(a - 3r)v^2 + a + r)}{(1 + u^2)^2(v^2 + 1)^2} \\ &= x(u, v). \end{aligned}$$

Then, for any u, we have $x(u,v) = x(u_0, v)$, where $u_0 \in [-1,1]$ is defined by

$$u_0 = \begin{cases} u & \text{if } u \in [-1,1], \\ 1/u & \text{if } u \notin [-1,1]. \end{cases}$$

Observe now that for $v \neq 0$

$$\begin{aligned} x(u,1/v) &= \frac{(u^4-6u^2+1)((a+r)(1/v)^4+2(a-3r)(1/v)^2+a+r)}{(u^2+1)^2((1/v)^2+1)^2} \\ &= \frac{(u^4-6u^2+1)((a+r)(1/v)^4+2(a-3r)(1/v)^2+a+r)}{(u^2+1)^2(1/v)^4(1+v^2)^2} \\ &= v^4\left(\frac{(u^4-6u^2+1)((a+r)(1/v)^4+2(a-3r)(1/v)^2+a+r)}{(u^2+1)^2(1+v^2)^2}\right) \\ &= \frac{(u^4-6u^2+1)((a+r)+2(a-3r)v^2+(a+r)v^4)}{(u^2+1)^2(1+v^2)^2} \\ &= x(u,v). \end{aligned}$$

Then, for any v, we have $x(u,v) = x(u, v_0)$, where $v_0 \in [-1,1]$ is defined by

$$v_0 = \begin{cases} v & \text{if } v \in [-1,1], \\ 1/v & \text{if } v \notin [-1,1]. \end{cases}$$

For y we have

$$\begin{aligned} y(1/u,v) &= \frac{4(1/u)(1-(1/u)^2)((a+r)v^4+2(a-3r)v^2+a+r)}{((1/u)^2+1)^2(v^2+1)^2} \\ &= \frac{4(1/u)(1-(1/u)^2)((a+r)v^4+2(a-3r)v^2+a+r)}{(1/u)^4(1+u^2)^2(v^2+1)^2} \\ &= u^4\left(\frac{4(1/u)(1-(1/u)^2)((a+r)v^4+2(a-3r)v^2+a+r)}{(1+u^2)^2(v^2+1)^2}\right) \\ &= \frac{4u(u^2-1)^2((a+r)v^4+2(a-3r)v^2+a+r)}{(1/u)^4(1+u^2)^2(v^2+1)^2} \\ &= \frac{4(-u)(1-u^2)^2((a+r)v^4+2(a-3r)v^2+a+r)}{(1/u)^4(1+u^2)^2(v^2+1)^2} \\ &= y(-u,v). \end{aligned}$$

Then, for any u, we have $y(u,v) = y(u_0, v)$, where $u_0 \in [-1,1]$ is defined by

$$u_0 = \begin{cases} u & \text{if } u \in [-1,1], \\ -1/u & \text{if } u \notin [-1,1]. \end{cases}$$

Also,

$$\begin{aligned}
y(u,1/v) &= \frac{4u(1-u^2)((a+r)(1/v)^4+2(a-3r)(1/v)^2+a+r)}{(u^2+1)^2((1/v)^2+1)^2} \\
&= \frac{4u(1-u^2)((a+r)(1/v)^4+2(a-3r)(1/v)^2+a+r)}{(u^2+1)^2(1/v)^4(1+v^2)^2} \\
&= v^4\left(\frac{4u(1-u^2)((a+r)(1/v)^4+2(a-3r)(1/v)^2+a+r)}{(u^2+1)^2(1+v^2)^2}\right) \\
&= \frac{4u(1-u^2)((a+r)+2(a-3r)v^2+(a+r)v^2)}{(u^2+1)^2(1+v^2)^2} \\
&= y(u,v).
\end{aligned}$$

Then, for any v, we have $y(u,v)=y(u,v_0)$, where $v_0\in[-1,1]$ is defined by

$$v_0 = \begin{cases} v & \text{if } v\in[-1,1], \\ 1/v & \text{if } v\notin[-1,1]. \end{cases}$$

And finally,

$$\begin{aligned}
z(1/u,v) &= \frac{8r(1/u)v(1+(1/u)^2)(1-v^2)}{((1/u)^2+1)^2(v^2+1)^2} \\
&= \frac{8r(1/u)v(1+(1/u)^2)(1-v^2)}{(1/u)^4(1+u^2)^2(v^2+1)^2} \\
&= u^4\left(\frac{8r(1/u)v(1+(1/u)^2)(1-v^2)}{(1+u^2)^2(v^2+1)^2}\right) \\
&= \frac{8ruv(u^2+1)(1-v^2)}{(1+u^2)^2(v^2+1)^2} \\
&= z(u,v).
\end{aligned}$$

Then, for any u, we have $z(u,v)=z(u_0,v)$, where $u_0\in[-1,1]$ is defined by

$$u_0 = \begin{cases} u & \text{if } u\in[-1,1], \\ 1/u & \text{if } u\notin[-1,1]. \end{cases}$$

And,

$$\begin{aligned}
z(u,1/v) &= \frac{8ru(1/v)(1+u^2)(1-(1/v)^2)}{(u^2+1)^2((1/v)^2+1)^2} \\
&= \frac{8ru(1/v)(1+u^2)(1-(1/v)^2)}{(u^2+1)^2(1/v)^4(1+v^2)^2} \\
&= v^4\left(\frac{8ru(1/v)(1+u^2)(1-(1/v)^2)}{(u^2+1)^2(1+v^2)^2}\right) \\
&= \frac{8ruv(1+u^2)(v^2-1)}{(u^2+1)^2(1+v^2)^2} \\
&= \frac{8ru(-v)(1+u^2)(1-v^2)}{(u^2+1)^2(1+v^2)^2} \\
&= z(u,-v).
\end{aligned}$$

Then, for any v, we have $z(u,v) = z(u,v_0)$, where $v_0 \in [-1,1]$ is defined by

$$v_0 = \begin{cases} v & \text{if } v \in [-1,1], \\ -1/v & \text{if } v \notin [-1,1]. \end{cases}$$

With this we conclude that the entire surface is obtained for $u, v \in [-1,1]$

A plot of this surface using $a = 2$ and $r = 1$ can be seen in Figure 4.

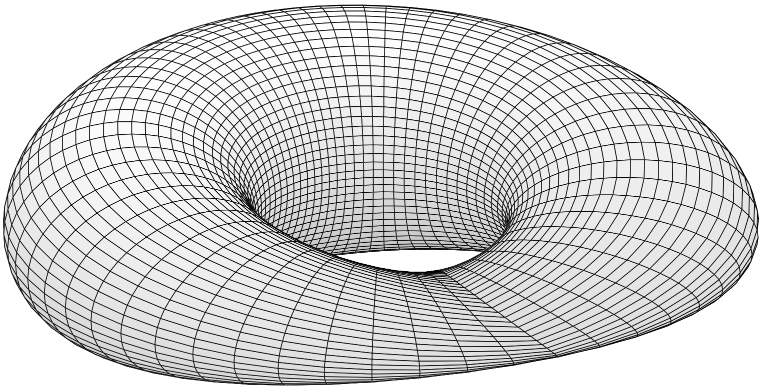


Figure 4: Plot of the surface.

A plot of this surface using $u \in [-1,1]$ and $v \in [0,1]$ can be seen in Figure 5.

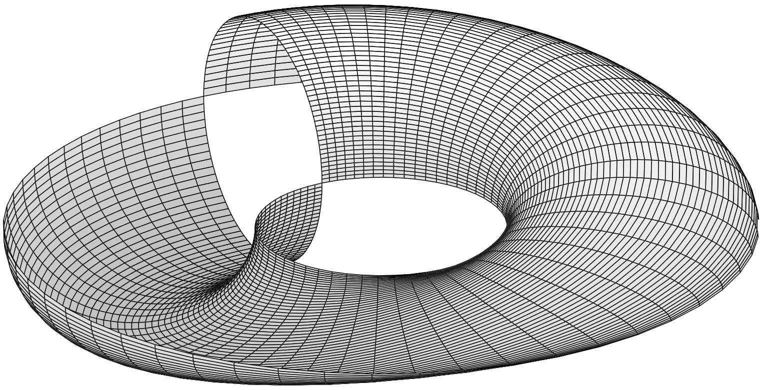


Figure 5: Plot of the surface using $u \in [-1,1]$ and $v \in [0,1]$.

It is easy to see that for $u \in [-1,1]$

$$\begin{aligned} x(u,1) &= x(u,-1) \\ y(u,1) &= y(u,-1) \\ z(u,1) &= 0 = z(u,-1), \end{aligned}$$

so the points of this surface at $(u,1)$ are glued at the points $(u,-1)$.

Notice also that for $v \in [-1,1]$

$$\begin{aligned} x(1,v) &= x(-1,-v) \\ y(1,v) &= 0 = y(-1,-v) \\ z(1,v) &= z(-1,-v), \end{aligned}$$

then the points of this surface at $(1,v)$ are glued at the points $(-1,-v)$. Therefore we see that this surface is a Klein bottle (projected in $\mathbb{R}^3$).

Problem 4

(a) Consider the map $\mathcal{H} : \mathbb{R}^3 \to \mathbb{R}^4$ defined such that

$$(x, y, z) \mapsto (xy, yz, xz, x^2 - y^2).$$

Let (x, y, z) and $(x', y', z') \in S^2$ (in $\mathbb{R}^3$). If $(x', y', z') = (-x, -y, -z)$ then

$$\begin{aligned}\mathcal{H}(x', y', z') &= \mathcal{H}(-x, -y, -z)\\ &= ((-x)(-y), (-y)(-z), (-x)^2 - (-y)^2)\\ &= (xy, yz, xz, x^2 - y^2)\\ &= \mathcal{H}(x, y, z).\end{aligned}$$

So antipodal points have the same image.

Let's check the opposite: if the image of two points are the same, then they are the same or they are antipodals.

Let (x, y, z) and $(x', y', z') \in S^2$ (in $\mathbb{R}^3$) such that $\mathcal{H}(x, y, z) = \mathcal{H}(x', y', z')$, then

$$xy = x'y' \tag{1}$$
$$yz = y'z' \tag{2}$$
$$xz = x'z' \tag{3}$$
$$x^2 - y^2 = x'^2 - y'^2. \tag{4}$$

Summing the square of (2) and of (3):

$$\begin{aligned}y^2z^2 + x^2z^2 &= y'^2z'^2 + x'^2z'^2\\ (y^2 + x^2)z^2 &= (y'^2 + x'^2)z'^2\\ (y^2 + x^2)(1 - x^2 - y^2) &= (y'^2 + x'^2)(1 - x'^2 - y'^2)\\ (y^2 + x^2) - x^2(y^2 + x^2) - y^2(y^2 + x^2) &= (y'^2 + x'^2) - x'^2(y'^2 + x'^2) - y'^2(y'^2 + x'^2)\\ y^2 + x^2 - x^2y^2 - x^4 - y^4 - x^2y^2 &= y'^2 + x'^2 - x'^2y'^2 - x'^4 - y'^4 - x'^2y'^2\\ y^2 + x^2 - x^4 - y^4 &= y'^2 + x'^2 - x'^4 - y'^4.\end{aligned}$$

Observe that, from (4):

$$\begin{aligned}(x^2 - y^2)^2 &= (x'^2 - y'^2)^2\\ x^4 - 2x^2y^2 + y^4 &= x'^4 - 2x'^2y'^2 + y'^4\\ x^4 + y^4 &= x'^4 + y'^4.\end{aligned}$$

So, using this and the previous equation we obtain

$$y^2 + x^2 = y'^2 + x'^2. \tag{5}$$

Summing (4) and (5) we get:

$$x^2 = x'^2.$$

Subtracting (4) and (5) we get:

$$y^2 = y'^2.$$

And finally

$$z^2 = 1 - x^2 - y^2 = 1 - x'^2 - y'^2 = z'^2.$$

Them we have $x' = \pm x$, $y' = \pm y$ and $z' = \pm z$. This gives us 8 possibilities

$$\begin{aligned}
(x', y', z') &= (x, y, z) \\
(x', y', z') &= (-x, -y, -z) \\
(x', y', z') &= (-x, y, z) \\
(x', y', z') &= (x, -y, -z) \\
(x', y', z') &= (x, -y, z) \\
(x', y', z') &= (-x, y, -z) \\
(x', y', z') &= (x, y, -z) \\
(x', y', z') &= (-x, -y, z).
\end{aligned}$$

We already know that the first two cases satisfy $\mathcal{H}(x, y, z) = \mathcal{H}(x', y', z')$, let's check if some other case also satisfies.

As $(-x, y, z)$ and $(x, -y, -z)$ are antipodal points, then we know that $\mathcal{H}(-x, y, z) = \mathcal{H}(x, -y, -z)$, then it is suficient to check if one of them has the same image as (x, y, z). So let's check if $\mathcal{H}(x, y, z) = \mathcal{H}(-x, y, z)$.

$$\begin{aligned}
\mathcal{H}(x, y, z) &= \mathcal{H}(-x, y, z) \\
(xy, yz, xz, x^2 - y^2) &= (-xy, yz, -xz, x^2 - y^2),
\end{aligned}$$

so $xy = -xy$ and $xz = -xz$. If $x = 0$, both conditions are satisfied, but in this case $(x', y', z') = (-x, y, z) = (0, y, z) = (x, y, z)$, so actually they are the same point. If $x \neq 0$, the conditions imply that $y = 0$ and $z = 0$, so $x = \pm 1$, but in this case the points are antipodal: $(1, 0, 0)$ and $(-1, 0, 0)$.

As $(x, -y, z)$ and $(-x, y, -z)$ are antipodal points, then it is suficient to check if one of them has the same image as (x, y, z). So let's check if $\mathcal{H}(x, y, z) = \mathcal{H}(x, -y, z)$.

$$\begin{aligned}
\mathcal{H}(x, y, z) &= \mathcal{H}(x, -y, z) \\
(xy, yz, xz, x^2 - y^2) &= (-xy, -yz, xz, x^2 - y^2),
\end{aligned}$$

so $xy = -xy$ and $yz = -yz$. If $y = 0$, both conditions are satisfied, but in this case $(x', y', z') = (x, -y, z) = (x, 0, z) = (x, y, z)$, so actually they are the same point. If $y \neq 0$, the conditions imply that $x = 0$ and $z = 0$, so $y = \pm 1$, but in this case the points are antipodal: $(0, 1, 0)$ and $(0, -1, 0)$.

Finally the points $(x, y, -z)$ and $(-x, -y, z)$ are antipodal, then it is suficient to check if one of them has the same image as (x, y, z). So let's check if $\mathcal{H}(x, y, z) = \mathcal{H}(x, y, -z)$.

$$\begin{aligned}
\mathcal{H}(x, y, z) &= \mathcal{H}(x, y, -z) \\
(xy, yz, xz, x^2 - y^2) &= (xy, -yz, -xz, x^2 - y^2),
\end{aligned}$$

so $yz = -yz$ and $xz = -xz$. If $z = 0$, both conditions are satisfied, but in this case $(x', y', z') = (x, y, -z) = (x, y, 0) = (x, y, z)$, so actually they are the same point. If $z \neq 0$, the conditions imply that $x = 0$ and $y = 0$, so $z = \pm 1$, but in this case the points are antipodal: $(0, 0, 1)$ and $(0, 0, -1)$.

Then, for every case, if $\mathcal{H}(x, y, z) = \mathcal{H}(x', y', z')$ then $(x', y', z') = (x, y, z)$ or $(x', y', z') = (-x, -y, -z)$.

Now let $\tilde{\mathcal{H}} : \mathbb{RP}^2 \to \mathcal{H}(S^2)$ be the restriction of the map $\mathcal{H}$ to the projective plane. If $\tilde{\mathcal{H}}(x,y,z) = \tilde{\mathcal{H}}(x',y',z')$, we have just seen that in S^2 this happens iff $(x,y,z) = (x',y',z')$ or $(-x,-y,-z) = (x',y',z')$, but we know that in the projective plane there is a equivalence relation between pairs of antipodal points in S^2, so $\tilde{\mathcal{H}}(x,y,z) = \tilde{\mathcal{H}}(x',y',z')$ iff $(x,y,z) = (x',y',z')$, which means that $\tilde{\mathcal{H}}$ is injective onto $\tilde{\mathcal{H}}(\mathbb{RP}^2) = \mathcal{H}(S^2)$. The map $\tilde{\mathcal{H}}$ is continuous (simple polynomials), it is invertible in $\mathcal{H}(S^2)$ (as it is injective, and trivially surjective on it), and it is a homeomorphism, because it is a continuous injective map from a compact domain ($\mathbb{RP}^2$) onto its image in $\mathbb{R}^4$.

b) Consider the three maps from $\mathbb{R}^2$ to $\mathbb{R}^4$ given by

$$\psi_1(u,v) = \left(\frac{uv}{u^2+v^2+1}, \frac{v}{u^2+v^2+1}, \frac{u}{u^2+v^2+1}, \frac{u^2-v^2}{u^2+v^2+1}\right),$$
$$\psi_2(u,v) = \left(\frac{u}{u^2+v^2+1}, \frac{v}{u^2+v^2+1}, \frac{uv}{u^2+v^2+1}, \frac{u^2-1}{u^2+v^2+1}\right),$$
$$\psi_3(u,v) = \left(\frac{u}{u^2+v^2+1}, \frac{uv}{u^2+v^2+1}, \frac{v}{u^2+v^2+1}, \frac{1-u^2}{u^2+v^2+1}\right).$$

It is easy to see that ψ_1 is the composition $\mathcal{H} \circ \alpha_1$, where $\alpha_1 : \mathbb{R}^2 \to S^2$ is given by

$$\alpha_1(u,v) = \left(\frac{u}{\sqrt{u^2+v^2+1}}, \frac{v}{\sqrt{u^2+v^2+1}}, \frac{1}{\sqrt{u^2+v^2+1}}\right),$$

that ψ_1 is the composition $\mathcal{H} \circ \alpha_1$, where $\alpha_1 : \mathbb{R}^2 \to S^2$ is given by

$$\alpha_2(u,v) = \left(\frac{u}{\sqrt{u^2+v^2+1}}, \frac{1}{\sqrt{u^2+v^2+1}}, \frac{v}{\sqrt{u^2+v^2+1}}\right),$$

and that ψ_1 is the composition $\mathcal{H} \circ \alpha_1$, where $\alpha_1 : \mathbb{R}^2 \to S^2$ is given by

$$\alpha_3(u,v) = \left(\frac{1}{\sqrt{u^2+v^2+1}}, \frac{u}{\sqrt{u^2+v^2+1}}, \frac{v}{\sqrt{u^2+v^2+1}}\right).$$

Suppose that we have $\alpha_1(u_0,v_0) = \alpha_1(u_1,v_1)$, then

$$\frac{u_0}{\sqrt{u_0^2+v_0^2+1}} = \frac{u_1}{\sqrt{u_1^2+v_1^2+1}}$$
$$\frac{v_0}{\sqrt{u_0^2+v_0^2+1}} = \frac{v_1}{\sqrt{u_1^2+v_1^2+1}}$$
$$\frac{1}{\sqrt{u_0^2+v_0^2+1}} = \frac{1}{\sqrt{u_1^2+v_1^2+1}}.$$

Then

$$\frac{u_0}{\sqrt{u_0^2+v_0^2+1}} = \frac{u_1}{\sqrt{u_0^2+v_0^2+1}}$$
$$u_0 = u_1,$$

and

$$\frac{v_0}{\sqrt{u_0^2+v_0^2+1}} = \frac{v_1}{\sqrt{u_0^2+v_0^2+1}}$$
$$v_0 = v_1,$$

so α_1 is a injective map. The same happens for α_2 and α_3, as their coordinates are simple permutations of the coordinates of α_1. Then $\alpha_1, \alpha_2, \alpha_3$ are injective maps.

Now if $\psi_1(u_0, v_0) = \psi_1(u_1, v_1)$ then $\mathcal{H}(\alpha_1(u_0, v_0)) = \mathcal{H}(\alpha_1(u_1, v_1))$, which implies that $\alpha_1(u_0, v_0) = \alpha_1(u_1, v_1)$ or $\alpha_1(u_0, v_0) = -\alpha_1(u_1, v_1)$, the latter will never be satisfied, because the z coordinates of $\alpha_1(u_0, v_0)$ and $\alpha_1(u_1, v_1)$ are always greater than zero, so the former must be valid, but we saw that α_1 is injective, this implies that $(u_0, v_0) = (u_1, v_1)$, so ψ_1 is injective.

Similarly, we can prove that ψ_2 and ψ_3 are also injective maps, we just need to notice that $\alpha_2(u_0, v_0) = -\alpha_2(u_1, v_1)$ will never be satisfied, because the y coordinates of $\alpha_2(u_0, v_0)$ and $\alpha_2(u_1, v_1)$ are always greater than zero, so $\psi_2(u_0, v_0) = \psi_2(u_1, v_1)$ iff $(u_0, v_0) = (u_1, v_1)$, and that $\alpha_3(u_0, v_0) = -\alpha_3(u_1, v_1)$ will never be satisfied, because the y coordinates of $\alpha_3(u_0, v_0)$ and $\alpha_3(u_1, v_1)$ are always greater than zero, so $\psi_3(u_0, v_0) = \psi_3(u_1, v_1)$ iff $(u_0, v_0) = (u_1, v_1)$,

The term $u^2 + v^2 + 1$ is always greater than zero, so $\sqrt{u^2 + v^2 + 1}$ is continuous in $\mathbb{R}^2$, and it is also greater than zero, then $1/\sqrt{u^2 + v^2 + 1}$, $u/\sqrt{u^2 + v^2 + 1}$ and $v/\sqrt{u^2 + v^2 + 1}$ are all continuous functions, so α_1, α_2 and α_3 are continuous maps.

As $\mathcal{H}, \alpha_1, \alpha_2, \alpha_3$ are continuous, then ψ_1, ψ_2 and ψ_3 are compositions of continuous functions, then they are also continuous.

Now let's prove that each ψ_i is nonsingular. For this, we use the fact that

$$J\psi_i(u, v) = J\mathcal{H}(\alpha_i(u, v)) \cdot J\alpha_i(u, v),$$

where J is the Jacobian. We have

$$J\mathcal{H}(x, y, z) = \begin{pmatrix} \frac{\partial}{\partial x}(xy) & \frac{\partial}{\partial y}(xy) & \frac{\partial}{\partial z}(xy) \\ \frac{\partial}{\partial x}(yz) & \frac{\partial}{\partial y}(yz) & \frac{\partial}{\partial z}(yz) \\ \frac{\partial}{\partial x}(xz) & \frac{\partial}{\partial y}(xz) & \frac{\partial}{\partial z}(xz) \\ \frac{\partial}{\partial x}(x^2-y^2) & \frac{\partial}{\partial y}(x^2-y^2) & \frac{\partial}{\partial z}(x^2-y^2) \end{pmatrix} = \begin{pmatrix} y & x & 0 \\ 0 & z & y \\ z & 0 & x \\ 2x & -2y & 0 \end{pmatrix}.$$

$$J\mathcal{H}(\alpha_1(u, v)) = \begin{pmatrix} \frac{v}{\sqrt{u^2+v^2+1}} & \frac{u}{\sqrt{u^2+v^2+1}} & 0 \\ 0 & \frac{1}{\sqrt{u^2+v^2+1}} & \frac{v}{\sqrt{u^2+v^2+1}} \\ \frac{1}{\sqrt{u^2+v^2+1}} & 0 & \frac{u}{\sqrt{u^2+v^2+1}} \\ \frac{2u}{\sqrt{u^2+v^2+1}} & \frac{-2v}{\sqrt{u^2+v^2+1}} & 0 \end{pmatrix} = (u^2+v^2+1)^{-1/2} \begin{pmatrix} v & u & 0 \\ 0 & 1 & v \\ 1 & 0 & u \\ 2u & -2v & 0 \end{pmatrix}.$$

$$J\alpha_1(x, y, z) = \begin{pmatrix} \frac{\partial}{\partial u}\left(\frac{u}{\sqrt{u^2+v^2+1}}\right) & \frac{\partial}{\partial v}\left(\frac{u}{\sqrt{u^2+v^2+1}}\right) \\ \frac{\partial}{\partial u}\left(\frac{v}{\sqrt{u^2+v^2+1}}\right) & \frac{\partial}{\partial v}\left(\frac{v}{\sqrt{u^2+v^2+1}}\right) \\ \frac{\partial}{\partial u}\left(\frac{1}{\sqrt{u^2+v^2+1}}\right) & \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{u^2+v^2+1}}\right) \end{pmatrix}.$$

Where

$$\begin{aligned}
\frac{\partial}{\partial u}\left(\frac{u}{\sqrt{u^2+v^2+1}}\right) &= \frac{(u^2+v^2+1)^{1/2} - u(1/2)(u^2+v^2+1)^{-1/2}(2u)}{u^2+v^2+1} \\
&= \frac{((u^2+v^2+1)^{1/2}(u^2+v^2+1)^{1/2} - u^2)(u^2+v^2+1)^{-1/2}}{u^2+v^2+1} \\
&= \frac{u^2+v^2+1-u^2}{(u^2+v^2+1)^{3/2}} \\
&= \frac{v^2+1}{(u^2+v^2+1)^{3/2}},
\end{aligned}$$

$$\begin{aligned}
\frac{\partial}{\partial v}\left(\frac{u}{\sqrt{u^2+v^2+1}}\right) &= u(-1/2)(u^2+v^2+1)^{-3/2}(2v) \\
&= \frac{-uv}{(u^2+v^2+1)^{3/2}},
\end{aligned}$$

$$\frac{\partial}{\partial u}\left(\frac{v}{\sqrt{u^2+v^2+1}}\right) = \frac{-uv}{(u^2+v^2+1)^{3/2}},$$

$$\frac{\partial}{\partial v}\left(\frac{v}{\sqrt{u^2+v^2+1}}\right) = \frac{u^2+1}{(u^2+v^2+1)^{3/2}},$$

$$\begin{aligned}
\frac{\partial}{\partial u}\left(\frac{1}{\sqrt{u^2+v^2+1}}\right) &= (-1/2)(u^2+v^2+1)^{-3/2}(2u) \\
&= \frac{-u}{(u^2+v^2+1)^{3/2}},
\end{aligned}$$

$$\frac{\partial}{\partial v}\left(\frac{1}{\sqrt{u^2+v^2+1}}\right) = \frac{-v}{(u^2+v^2+1)^{3/2}}.$$

So

$$J\alpha_1(u,v) = \begin{pmatrix} \frac{v^2+1}{(u^2+v^2+1)^{3/2}} & \frac{-uv}{(u^2+v^2+1)^{3/2}} \\ \frac{-uv}{(u^2+v^2+1)^{3/2}} & \frac{u^2+1}{(u^2+v^2+1)^{3/2}} \\ \frac{-u}{(u^2+v^2+1)^{3/2}} & \frac{-v}{(u^2+v^2+1)^{3/2}} \end{pmatrix} = (u^2+v^2+1)^{-3/2}\begin{pmatrix} v^2+1 & -uv \\ -uv & u^2+1 \\ -u & -v \end{pmatrix}.$$

Then,

$$J\psi_1(u,v) = (u^2+v^2+1)^{-2}\begin{pmatrix} v & u & 0 \\ 0 & 1 & v \\ 1 & 0 & u \\ 2u & -2v & 0 \end{pmatrix}\begin{pmatrix} v^2+1 & -uv \\ -uv & u^2+1 \\ -u & -v \end{pmatrix}$$

$$J\psi_1(u,v) = (u^2+v^2+1)^{-2}\begin{pmatrix} v^3+v-u^2v & -uv^2+u^3+u \\ -2uv & u^2+1-v^2 \\ v^2+1-u^2 & -2uv \\ 4uv^2+2u & -4u^2v-2v \end{pmatrix}$$

To prove that $\psi_1(u,v)$ is nonsingular, it is sufficient to prove that one of its submatrices 2 by 2 has non zero determinant. Observe the submatrix formed with the second and third rows:

$$A_{23} = (u^2+v^2+1)^{-2}\begin{pmatrix} -2uv & 1+(u^2-v^2) \\ 1-(u^2-v^2) & -2uv \end{pmatrix}$$

The determinant of A_{23} is

$$\begin{aligned} |A_{23}| &= (u^2+v^2+1)^{-4}(4u^2v^2-1+(u^2-v^2)^2) \\ &= (u^2+v^2+1)^{-4}(4u^2v^2-1+u^4-2u^2v^2+v^4) \\ &= (u^2+v^2+1)^{-4}(-1+u^4+2u^2v^2+v^4) \\ &= (u^2+v^2+1)^{-4}(-1+(u^2+v^2)^2), \end{aligned}$$

this will be non-zero when $u^2+v^2 \neq 1$. If $u^2+v^2=1$, we can check the submatrix A_{14}:

$$A_{14} = (u^2+v^2+1)^{-2}\begin{pmatrix} v^3+v-u^2v & -uv^2+u^3+u \\ 4uv^2+2u & -4u^2v-2v \end{pmatrix} = \frac{1}{4}\begin{pmatrix} v^3+v-u^2v & -uv^2+u^3+u \\ 4uv^2+2u & -4u^2v-2v \end{pmatrix}$$

whose determinant is

$$\begin{aligned} |A_{14}| &= \frac{1}{16}\big((v^3+v-u^2v)(-4u^2v-2v)-(4uv^2+2u)(-uv^2+u^3+u)\big) \\ &= \frac{1}{16}\big((v^3+v-u^2v)(-4u^2v)-(v^3+v-u^2v)(2v) \\ &\quad -(4uv^2+2u)(-uv^2)-(4uv^2+2u)u^3-(4uv^2+2u)u\big) \\ &= \frac{1}{16}\big(-4u^2v^4-4u^2v^2+4u^4v^2-2v^4-2v^2+2u^2v^2 \\ &\quad +4u^2v^4+2u^2v^2-4u^4v^2-2u^4-4u^2v^2-2u^2\big) \\ &= \frac{1}{16}\big(-4u^2v^2-2v^4-2v^2-2u^4-2u^2\big) \\ &= \frac{1}{16}\big(-2((u^2+v^2)^2+u^2+v^2)\big) \\ &= \frac{1}{16}\big(-2(1+1)\big) \\ &= \frac{1}{16}\big(-4\big) \\ &= -1/4. \end{aligned}$$

So, for every $(u,v)\in\mathbb{R}^2$, $J\psi_1$ is nonsingular, therefore ψ_1 is nonsingular.

For ψ_2 we have

$$J\psi_2(u,v) = J\mathcal{H}(\alpha_2(u,v))\cdot J\alpha_2(u,v),$$

where

$$J\mathcal{H}(\alpha_2(u,v)) = \begin{pmatrix} \frac{1}{\sqrt{u^2+v^2+1}} & \frac{u}{\sqrt{u^2+v^2+1}} & 0 \\ 0 & \frac{v}{\sqrt{u^2+v^2+1}} & \frac{1}{\sqrt{u^2+v^2+1}} \\ \frac{v}{\sqrt{u^2+v^2+1}} & 0 & \frac{u}{\sqrt{u^2+v^2+1}} \\ \frac{2u}{\sqrt{u^2+v^2+1}} & \frac{-2}{\sqrt{u^2+v^2+1}} & 0 \end{pmatrix}$$

$$J\mathcal{H}(\alpha_2(u,v)) = \frac{1}{\sqrt{u^2+v^2+1}} \begin{pmatrix} 1 & u & 0 \\ 0 & v & 1 \\ v & 0 & u \\ 2u & -2 & 0 \end{pmatrix}.$$

And

$$J\alpha_2(x,y,z) = \begin{pmatrix} \frac{\partial}{\partial u}\left(\frac{u}{\sqrt{u^2+v^2+1}}\right) & \frac{\partial}{\partial v}\left(\frac{u}{\sqrt{u^2+v^2+1}}\right) \\ \frac{\partial}{\partial u}\left(\frac{1}{\sqrt{u^2+v^2+1}}\right) & \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{u^2+v^2+1}}\right) \\ \frac{\partial}{\partial u}\left(\frac{v}{\sqrt{u^2+v^2+1}}\right) & \frac{\partial}{\partial v}\left(\frac{v}{\sqrt{u^2+v^2+1}}\right) \end{pmatrix}.$$

$$J\alpha_2(u,v) = \begin{pmatrix} \frac{v^2+1}{(u^2+v^2+1)^{3/2}} & \frac{-uv}{(u^2+v^2+1)^{3/2}} \\ \frac{-u}{(u^2+v^2+1)^{3/2}} & \frac{-v}{(u^2+v^2+1)^{3/2}} \\ \frac{-uv}{(u^2+v^2+1)^{3/2}} & \frac{u^2+1}{(u^2+v^2+1)^{3/2}} \end{pmatrix} = (u^2+v^2+1)^{-3/2} \begin{pmatrix} v^2+1 & -uv \\ -u & -v \\ -uv & u^2+1 \end{pmatrix}.$$

Then,

$$J\psi_2(u,v) = (u^2+v^2+1)^{-2} \begin{pmatrix} 1 & u & 0 \\ 0 & v & 1 \\ v & 0 & u \\ 2u & -2 & 0 \end{pmatrix} \begin{pmatrix} v^2+1 & -uv \\ -u & -v \\ -uv & u^2+1 \end{pmatrix}$$

$$J\psi_2(u,v) = (u^2+v^2+1)^{-2} \begin{pmatrix} v^2+1-u^2 & -2uv \\ -2uv & -v^2+u^2+1 \\ v^3+v-u^2v & -uv^2+u^3+u \\ 2uv^2+4u & -2u^2v+2v \end{pmatrix}$$

Observe the submatrix formed with the first and second rows:

$$A_{12} = (u^2+v^2+1)^{-2} \begin{pmatrix} 1-(u^2-v^2) & -2uv \\ -2uv & 1+(u^2-v^2) \end{pmatrix}$$

The determinant of A_{12} is

$$\begin{aligned} |A_{12}| &= (u^2+v^2+1)^{-4}(1-(u^2-v^2)^2-4u^2v^2) \\ &= (u^2+v^2+1)^{-4}(1-(u^4-2u^2v^2+v^4)-4u^2v^2) \\ &= (u^2+v^2+1)^{-4}(1-u^4-2u^2v^2-v^4) \\ &= (u^2+v^2+1)^{-4}(1-(u^2+v^2)^2), \end{aligned}$$

which is non-zero when $u^2+v^2 \neq 1$.

If $u^2+v^2=1$, we can take the submatrix A_{14}:

$$A_{14} = \frac{1}{4} \begin{pmatrix} 1-(u^2-v^2) & -2uv \\ 2uv^2+4u & -2u^2v+2v \end{pmatrix},$$

whose determinant is

$$\begin{aligned} |A_{14}| &= \frac{1}{16}\big(-2u^2v+2v-(u^2-v^2)(-2u^2v+2v)+4u^2v^3+8u^2v\big) \\ &= \frac{1}{16}\big(-2u^2v+2v+2u^4v-2u^2v-2u^2v^3+2v^3+4u^2v^3+8u^2v\big) \\ &= \frac{1}{16}\big(4u^2v+2v+2u^4v+2u^2v^3+2v^3\big) \\ &= \frac{1}{16}\big(2v(2u^2+1+u^4+u^2v^2+v^2)\big), \end{aligned}$$

which is zero only when $v = 0$, so $u^2 = 1$. But we can analyze the matrix A_{24} in the points $(\pm 1, 0)$:

$$A_{24} = \frac{1}{4}\begin{pmatrix} -2uv & -v^2+u^2+1 \\ 2uv^2+4u & -2u^2v+2v \end{pmatrix} = \begin{pmatrix} 0 & 0.5 \\ \pm 1 & 0 \end{pmatrix},$$

whose determinants are $\pm 0.5 \neq 0$.

Then, for every (u, v) there is a submatrix whose determinant is not zero, so ψ_2 is nonsingular.

Now let's check if ψ_3 is nonsingular.

$$J\psi_3(u,v) = J\mathcal{H}(\alpha_3(u,v)) \cdot J\alpha_3(u,v),$$

where

$$J\mathcal{H}(\alpha_3(u,v)) = \begin{pmatrix} \frac{u}{\sqrt{u^2+v^2+1}} & \frac{1}{\sqrt{u^2+v^2+1}} & 0 \\ 0 & \frac{v}{\sqrt{u^2+v^2+1}} & \frac{u}{\sqrt{u^2+v^2+1}} \\ \frac{v}{\sqrt{u^2+v^2+1}} & 0 & \frac{1}{\sqrt{u^2+v^2+1}} \\ \frac{2}{\sqrt{u^2+v^2+1}} & \frac{-2u}{\sqrt{u^2+v^2+1}} & 0 \end{pmatrix}$$

$$J\mathcal{H}(\alpha_3(u,v)) = \frac{1}{\sqrt{u^2+v^2+1}}\begin{pmatrix} u & 1 & 0 \\ 0 & v & u \\ v & 0 & 1 \\ 2 & -2u & 0 \end{pmatrix}.$$

And

$$J\alpha_3(x,y,z) = \begin{pmatrix} \frac{\partial}{\partial u}\left(\frac{1}{\sqrt{u^2+v^2+1}}\right) & \frac{\partial}{\partial v}\left(\frac{1}{\sqrt{u^2+v^2+1}}\right) \\ \frac{\partial}{\partial u}\left(\frac{u}{\sqrt{u^2+v^2+1}}\right) & \frac{\partial}{\partial v}\left(\frac{u}{\sqrt{u^2+v^2+1}}\right) \\ \frac{\partial}{\partial u}\left(\frac{v}{\sqrt{u^2+v^2+1}}\right) & \frac{\partial}{\partial v}\left(\frac{v}{\sqrt{u^2+v^2+1}}\right) \end{pmatrix}.$$

$$J\alpha_3(u,v) = \begin{pmatrix} \frac{-u}{(u^2+v^2+1)^{3/2}} & \frac{-v}{(u^2+v^2+1)^{3/2}} \\ \frac{v^2+1}{(u^2+v^2+1)^{3/2}} & \frac{-uv}{(u^2+v^2+1)^{3/2}} \\ \frac{-uv}{(u^2+v^2+1)^{3/2}} & \frac{u^2+1}{(u^2+v^2+1)^{3/2}} \end{pmatrix} = (u^2+v^2+1)^{-3/2}\begin{pmatrix} -u & -v \\ v^2+1 & -uv \\ -uv & u^2+1 \end{pmatrix}.$$

Then,

$$J\psi_3(u,v) = (u^2+v^2+1)^{-2}\begin{pmatrix} u & 1 & 0 \\ 0 & v & u \\ v & 0 & 1 \\ 2 & -2u & 0 \end{pmatrix}\begin{pmatrix} -u & -v \\ v^2+1 & -uv \\ -uv & u^2+1 \end{pmatrix}$$

$$J\psi_3(u,v) = (u^2+v^2+1)^{-2}\begin{pmatrix} -u^2+v^2+1 & -2uv \\ v^3+v-u^2v & -uv^2+u^3+u \\ -2uv & -v^2+u^2+1 \\ -4u-2uv^2 & -2v-2u^2v \end{pmatrix}$$

Observe the submatrix formed with the first and third rows:

$$A_{13} = (u^2+v^2+1)^{-2}\begin{pmatrix} 1-(u^2-v^2) & -2uv \\ -2uv & 1+(u^2-v^2) \end{pmatrix},$$

whose determinant $|A_{13}| = (u^2+v^2+1)^{-4}(1-(u^2+v^2)^2)$ is not zero when $u^2+v^2 \neq 1$. If $u^2+v^2 = 1$, we can take the submatrix A_{14}:

$$A_{14} = \frac{1}{4}\begin{pmatrix} 1-(u^2-v^2) & -2uv \\ -2uv^2-4u & -2u^2v-2v \end{pmatrix},$$

whose determinant is

$$\begin{aligned}
|A_{14}| &= \frac{1}{16}\big(-2u^2v-2v-(u^2-v^2)(-2u^2v-2v)-4u^2v^3-8u^2v\big) \\
&= \frac{1}{16}\big(-2u^2v-2v+2u^4v+2u^2v-2u^2v^3-2v^3-4u^2v^3-8u^2v\big) \\
&= \frac{1}{16}\big(8u^2v+2v+2u^4v-6u^2v^3+2v^3\big) \\
&= \frac{1}{16}\big(2v(4u^2+1+u^4-3u^2v^2+v^2)\big),
\end{aligned}$$

which is zero only when $v=0$, so $u^2=1$. But we can analyze the matrix A_{34} in the points $(\pm 1, 0)$:

$$A_{24} = \frac{1}{4}\begin{pmatrix} -2uv & -v^2+u^2+1 \\ -2uv^2-4u & -2u^2v-2v \end{pmatrix} = \begin{pmatrix} 0 & 0.5 \\ \pm 1 & 0 \end{pmatrix},$$

whose determinants are $\pm 0.5 \neq 0$. Then, for every (u,v) there is a submatrix whose determinant is not zero, so ψ_3 is nonsingular.

c) Let $\psi_1(u,v) = (x,y,z,t)$, then

$$\begin{aligned}
y^2+z^2 &= \left(\frac{v}{u^2+v^2+1}\right)^2 + \left(\frac{u}{u^2+v^2+1}\right)^2 \\
&= \frac{u^2+v^2}{(u^2+v^2+1)^2}.
\end{aligned}$$

If $u^2+v^2=1$ then

$$y^2+z^2 = \frac{1}{(1+1)^2} = \frac{1}{4}.$$

On the other hand, if $y^2+z^2 = 1/4$ then

$$\begin{aligned}
\frac{u^2+v^2}{(u^2+v^2+1)^2} &= 1/4 \\
4(u^2+v^2) &= (u^2+v^2+1)^2 \\
4(u^2+v^2) &= ((u^2+v^2)^2+2(u^2+v^2)+1) \\
0 &= ((u^2+v^2)^2-2(u^2+v^2)+1) \\
0 &= ((u^2+v^2)^2-1)^2 \\
0 &= (u^2+v^2)^2-1 \\
(u^2+v^2)^2 &= 1 \\
u^2+v^2 &= 1.
\end{aligned}$$

So $y^2 + z^2 = 1/4$ iff $u^2 + v^2 = 1$.

We will now prove that $y^2 + z^2 \leq 1/4$. First observe that if $u^2 + v^2 = 0$, then $y^2 + z^2 = 0 \leq 1/4$. Consider now $u^2 + v^2 \neq 0$, and define $k := u^2 + v^2$ (so $k > 0$), then we have

$$y^2 + z^2 = \frac{u^2 + v^2}{(u^2 + v^2 + 1)^2} = \frac{k}{(k+1)^2}.$$

Now define $\gamma(k) := \frac{(k+1)^2}{k}$, observe that $\gamma(k) = \frac{k^2+2k+1}{k} = k + 2 + 1/k$, then

$$\gamma'(k) = 1 - 1/k^2,$$

which is 0 for $k = 1$, so 1 is a critical point of γ. Observe that the second derivative

$$\gamma''(k) = 2/k^3$$

is positive for every $k > 0$, then 1 is a minimum point of γ, so $\gamma(k) \geq \gamma(1) = 4$. Then $y^2 + z^2 = 1/\gamma(k) \leq 1/4$.

We can see that u and v are solutions of the equations

$$(y^2 + z^2)u^2 - zu + z^2 = 0$$

$$(y^2 + z^2)v^2 - yv + y^2 = 0.$$

In fact, using the definition of y and z:

$$\begin{aligned}
(y^2 + z^2)u^2 - zu + z^2 &= \left(\frac{u^2 + v^2}{(u^2 + v^2 + 1)^2}\right) u^2 - \left(\frac{u}{u^2 + v^2 + 1}\right) u + \left(\frac{u}{u^2 + v^2 + 1}\right)^2 \\
&= \frac{u^2(u^2 + v^2)}{(u^2 + v^2 + 1)^2} - \frac{u^2}{u^2 + v^2 + 1} + \frac{u^2}{(u^2 + v^2 + 1)^2} \\
&= \frac{u^2(u^2 + v^2)}{(u^2 + v^2 + 1)^2} - \frac{u^2(u^2 + v^2 + 1)}{(u^2 + v^2 + 1)^2} + \frac{u^2}{(u^2 + v^2 + 1)^2} \\
&= \frac{u^2(u^2 + v^2)}{(u^2 + v^2 + 1)^2} - \frac{u^2(u^2 + v^2) + u^2}{(u^2 + v^2 + 1)^2} + \frac{u^2}{(u^2 + v^2 + 1)^2} \\
&= \frac{u^2(u^2 + v^2)}{(u^2 + v^2 + 1)^2} - \frac{u^2(u^2 + v^2)}{(u^2 + v^2 + 1)^2} - \frac{u^2}{(u^2 + v^2 + 1)^2} + \frac{u^2}{(u^2 + v^2 + 1)^2} \\
&= 0.
\end{aligned}$$

$$\begin{aligned}
(y^2 + z^2)v^2 - yv + y^2 &= \left(\frac{u^2 + v^2}{(u^2 + v^2 + 1)^2}\right) v^2 - \left(\frac{v}{u^2 + v^2 + 1}\right) v + \left(\frac{v}{u^2 + v^2 + 1}\right)^2 \\
&= \frac{v^2(u^2 + v^2)}{(u^2 + v^2 + 1)^2} - \frac{v^2}{u^2 + v^2 + 1} + \frac{v^2}{(u^2 + v^2 + 1)^2} \\
&= \frac{v^2(u^2 + v^2)}{(u^2 + v^2 + 1)^2} - \frac{v^2(u^2 + v^2 + 1)}{(u^2 + v^2 + 1)^2} + \frac{v^2}{(u^2 + v^2 + 1)^2} \\
&= \frac{v^2(u^2 + v^2)}{(u^2 + v^2 + 1)^2} - \frac{v^2(u^2 + v^2) + v^2}{(u^2 + v^2 + 1)^2} + \frac{v^2}{(u^2 + v^2 + 1)^2} \\
&= \frac{v^2(u^2 + v^2)}{(u^2 + v^2 + 1)^2} - \frac{v^2(u^2 + v^2)}{(u^2 + v^2 + 1)^2} - \frac{v^2}{(u^2 + v^2 + 1)^2} + \frac{v^2}{(u^2 + v^2 + 1)^2} \\
&= 0.
\end{aligned}$$

If we solve the equation $(y^2+z^2)u^2 - zu + z^2 = 0$, considering $y^2+z^2 \neq 0$, we find:

$$\begin{aligned}
u &= \frac{z \pm \sqrt{z^2 - 4(y^2+z^2)z^2}}{2(y^2+z^2)} \\
&= \frac{z \pm |z|\sqrt{1-4(y^2+z^2)}}{2(y^2+z^2)} \qquad \text{(always in } \mathbb{R} \text{ because } y^2+z^2 \leq 1/4) \\
&= \frac{z \pm z\sqrt{1-4(y^2+z^2)}}{2(y^2+z^2)} \qquad \text{(as we are using both signs for } z) \\
&= \frac{z\left(1 \pm \sqrt{1-4(y^2+z^2)}\right)}{2(y^2+z^2)}.
\end{aligned}$$

So we have two possibilities:

$$u_0 = \frac{z\left(1 - \sqrt{1-4(y^2+z^2)}\right)}{2(y^2+z^2)},$$

$$u_1 = \frac{z\left(1 + \sqrt{1-4(y^2+z^2)}\right)}{2(y^2+z^2)}.$$

Let's find out which one of them is u. First, observe that u_0:

$$\begin{aligned}
u_0 &= \frac{z\left(1 - \sqrt{1-4(y^2+z^2)}\right)}{2(y^2+z^2)} \\
&= \frac{\left(\frac{u}{u^2+v^2+1}\right)\left(1 - \sqrt{1-4\left(\frac{u^2+v^2}{(u^2+v^2+1)^2}\right)}\right)}{2\left(\frac{u^2+v^2}{(u^2+v^2+1)^2}\right)} \\
&= \left(\frac{u}{u^2+v^2+1}\right)\left(\frac{(u^2+v^2+1)^2}{u^2+v^2}\right)\frac{1}{2}\left(1 - \sqrt{1-4\left(\frac{u^2+v^2}{(u^2+v^2+1)^2}\right)}\right) \\
&= u\left(\frac{u^2+v^2+1}{u^2+v^2}\right)\frac{1}{2}\left(1 - \sqrt{1-4\left(\frac{u^2+v^2}{(u^2+v^2+1)^2}\right)}\right) \\
&= u\left(\frac{u^2+v^2+1}{u^2+v^2}\right)\frac{1}{2}\left(1 - \sqrt{\frac{(u^2+v^2+1)^2 - 4(u^2+v^2)}{(u^2+v^2+1)^2}}\right) \\
&= u\left(\frac{u^2+v^2+1}{u^2+v^2}\right)\frac{1}{2}\left(1 - \frac{\sqrt{(u^2+v^2)^2 + 2(u^2+v^2) + 1 - 4(u^2+v^2)}}{(u^2+v^2+1)^2}\right) \\
&= u\left(\frac{u^2+v^2+1}{u^2+v^2}\right)\frac{1}{2}\left(1 - \frac{\sqrt{(u^2+v^2)^2 - 2(u^2+v^2) + 1}}{u^2+v^2+1}\right) \\
&= u\left(\frac{u^2+v^2+1}{u^2+v^2}\right)\frac{1}{2}\left(1 - \frac{\sqrt{(u^2+v^2-1)^2}}{u^2+v^2+1}\right) \\
&= u\left(\frac{u^2+v^2+1}{u^2+v^2}\right)\frac{1}{2}\left(1 - \frac{|u^2+v^2-1|}{u^2+v^2+1}\right) \\
&= \left(\frac{u}{2(u^2+v^2)}\right)\left(u^2+v^2+1 - |u^2+v^2-1|\right).
\end{aligned}$$

And, similarly, we obtain

$$u_1 = \left(\frac{u}{2(u^2+v^2)}\right)\left(u^2+v^2+1+|u^2+v^2-1|\right).$$

If $u^2+v^2 \le 1$ then $u^2+v^2-1 \le 0$, so $|u^2+v^2-1| = -u^2-v^2+1$, then

$$\begin{aligned} u_0 &= \left(\frac{u}{2(u^2+v^2)}\right)\left(u^2+v^2+1-(-u^2-v^2+1)\right) \\ &= \left(\frac{u}{2(u^2+v^2)}\right)\left(u^2+v^2+1+u^2+v^2-1)\right) \\ &= \left(\frac{u}{2(u^2+v^2)}\right)\left(2(u^2+v^2)\right) \\ &= u. \end{aligned}$$

So, in this case, $u = u_0$.

If $u^2+v^2 \ge 1$ then $u^2+v^2-1 \ge 0$, so $|u^2+v^2-1| = u^2+v^2-1$, then

$$\begin{aligned} u_1 &= \left(\frac{u}{2(u^2+v^2)}\right)\left(u^2+v^2+1+(u^2+v^2-1)\right) \\ &= \left(\frac{u}{2(u^2+v^2)}\right)\left(2(u^2+v^2)\right) \\ &= u. \end{aligned}$$

So, in this case, $u = u_1$.

Now if we solve the equation $(y^2+z^2)v^2 - yv + y^2 = 0$, considering $y^2+z^2 \ne 0$, we find:

$$\begin{aligned} v &= \frac{y \pm \sqrt{y^2 - 4(y^2+z^2)y^2}}{2(y^2+z^2)} \\ &= \frac{y \pm |y|\sqrt{1-4(y^2+z^2)}}{2(y^2+z^2)} \\ &= \frac{y \pm y\sqrt{1-4(y^2+z^2)}}{2(y^2+z^2)} \\ &= \frac{y\left(1 \pm \sqrt{1-4(y^2+z^2)}\right)}{2(y^2+z^2)}. \end{aligned}$$

So we have two possibilities:

$$v_0 = \frac{y\left(1 - \sqrt{1-4(y^2+z^2)}\right)}{2(y^2+z^2)},$$

$$v_1 = \frac{y\left(1 + \sqrt{1-4(y^2+z^2)}\right)}{2(y^2+z^2)}.$$

Let's find out which one of them is v. First, observe that:

$$\begin{aligned}
v_0 &= \frac{y\left(1-\sqrt{1-4(y^2+z^2)}\right)}{2(y^2+z^2)} \\
&= \frac{\left(\frac{v}{u^2+v^2+1}\right)\left(1-\sqrt{1-4\left(\frac{u^2+v^2}{(u^2+v^2+1)^2}\right)}\right)}{2\left(\frac{u^2+v^2}{(u^2+v^2+1)^2}\right)} \\
&= \left(\frac{v}{2(u^2+v^2)}\right)\left(u^2+v^2+1-|u^2+v^2-1|\right).
\end{aligned}$$

And, similarly

$$v_1 = \left(\frac{v}{2(u^2+v^2)}\right)\left(u^2+v^2+1+|u^2+v^2-1|\right).$$

If $u^2+v^2 \leq 1$ then

$$\begin{aligned}
v_0 &= \left(\frac{v}{2(u^2+v^2)}\right)\left(u^2+v^2+1-(-u^2-v^2+1)\right) \\
&= \left(\frac{v}{2(u^2+v^2)}\right)\left(u^2+v^2+1+u^2+v^2-1)\right) \\
&= \left(\frac{v}{2(u^2+v^2)}\right)\left(2(u^2+v^2)\right) \\
&= v.
\end{aligned}$$

So, in this case, $v = v_0$.

If $u^2+v^2 \geq 1$ then

$$\begin{aligned}
v_1 &= \left(\frac{u}{2(u^2+v^2)}\right)\left(u^2+v^2+1+(u^2+v^2-1)\right) \\
&= \left(\frac{u}{2(u^2+v^2)}\right)\left(2(u^2+v^2)\right) \\
&= v.
\end{aligned}$$

So, in this case, $v = v_1$.

Observe also that when $y^2+z^2=0$, then $y=z=0$, but from the definition of ψ_1 this implies that $u=0$ and $v=0$.

So in short:

$$u = \begin{cases} 0 & \text{if } y=z=0; \\ \dfrac{z(1-\sqrt{1-4(y^2+z^2)})}{2(y^2+z^2)} & \text{if } u^2+v^2 \leq 1; \\ \dfrac{z(1+\sqrt{1-4(y^2+z^2)})}{2(y^2+z^2)} & \text{if } u^2+v^2 \geq 1. \end{cases}$$

$$v = \begin{cases} 0 & \text{if } y=z=0; \\ \dfrac{y(1-\sqrt{1-4(y^2+z^2)})}{2(y^2+z^2)} & \text{if } u^2+v^2 \leq 1; \\ \dfrac{y(1+\sqrt{1-4(y^2+z^2)})}{2(y^2+z^2)} & \text{if } u^2+v^2 \geq 1. \end{cases}$$

Given $(x, y, z, t) \in \psi_1(\mathbb{R}^2)$ we want to find which (u, v) such that $\psi_1(u, v) = (x, y, z, t)$, the conditions above use the values of u and v, let's find conditions based only on (x, y, z, t). First, let's make the change of variables to polar coordinates:

$$u = \rho \cos(\theta)$$

$$v = \rho \sin(\theta),$$

so we have

$$\psi_1(u, v) = \left(\frac{\rho^2 \cos\theta \sin\theta}{\rho^2 + 1}, \frac{\rho \sin\theta}{\rho^2 + 1}, \frac{\rho \cos\theta}{\rho^2 + 1}, \frac{\rho^2(\cos^2\theta - \sin^2\theta)}{\rho^2 + 1} \right).$$

Observe that

$$\begin{aligned} 2x &= \frac{2\rho^2 \cos\theta \sin\theta}{\rho^2 + 1} \\ &= \frac{\rho^2}{\rho^2 + 1} (2 \cos\theta \sin\theta) \\ &= \frac{\rho^2}{\rho^2 + 1} \sin 2\theta, \end{aligned}$$

and

$$\begin{aligned} t &= \frac{\rho^2(\cos^2\theta - \sin^2\theta)}{\rho^2 + 1} \\ &= \frac{\rho^2}{\rho^2 + 1} \cos 2\theta, \end{aligned}$$

so

$$(2x)^2 + t^2 = \frac{\rho^4}{(\rho^2 + 1)^2}.$$

The map $\rho \mapsto \frac{\rho^4}{(\rho^2+1)^2}$ is non-decrescent for $\rho \geq 0$, so if $\rho \leq 1$ then $\frac{\rho^4}{(\rho^2+1)^2} \leq 1/4$ and if $\rho \geq 1$ then $\frac{\rho^4}{(\rho^2+1)^2} \geq 1/4$. Observe that $\rho^2 = u^2 + v^2$, such that the condition $u^2 + v^2 \leq 1$ is equivalent to $\rho \leq 1$, which is equivalent to say $4x^2 + t^2 \leq 1/4$, as we have just seen.

Then we can write

$$u(x, y, z, t) = \begin{cases} 0 & \text{if } y = z = 0; \\ \dfrac{z(1 - \sqrt{1 - 4(y^2 + z^2)})}{2(y^2 + z^2)} & \text{if } 4x^2 + t^2 \leq 1/4 \\ \dfrac{z(1 + \sqrt{1 - 4(y^2 + z^2)})}{2(y^2 + z^2)} & \text{if } 4x^2 + t^2 \geq 1/4 \end{cases}$$

$$v(x, y, z, t) = \begin{cases} 0 & \text{if } y = z = 0; \\ \dfrac{y(1 - \sqrt{1 - 4(y^2 + z^2)})}{2(y^2 + z^2)} & \text{if } 4x^2 + t^2 \leq 1/4 \\ \dfrac{y(1 + \sqrt{1 - 4(y^2 + z^2)})}{2(y^2 + z^2)} & \text{if } 4x^2 + t^2 \geq 1/4. \end{cases}$$

Let's see that these expresions are continuous.

We know that, for $(x,y,z,t) \in \psi_1(\mathbb{R}^2)$ we have $0 \le y^2+z^2 \le 1/4$, so $1-4(y^2+z^2) \ge 0$, and then $\sqrt{1-4(y^2+z^2)}$ is continuous, and so the functions $z(1 \pm \sqrt{1-4(y^2+z^2)})$, $y(1 \pm \sqrt{1-4(y^2+z^2)})$ are continuous. Then the functions

$$\frac{z(1 \pm \sqrt{1-4(y^2+z^2)})}{2(y^2+z^2)},$$

$$\frac{y(1 \pm \sqrt{1-4(y^2+z^2)})}{2(y^2+z^2)}$$

are continuous when $y^2+z^2 \ne 0$. So each expression is continuous in its domain.

Now let a sequence $(x_n, y_n, z_n, t_n) \in \psi_1(\mathbb{R}^2)$, such that $y_n^2 + z_n^2 \ne 0$, $4x_n^2 + t_n^2 \le 1$, and (x_n, y_n, z_n, t_n) converge to $(0,0,0,0)$. Write y_n and z_n in polar coordinates so $y_n = \rho_n \cos\theta_n$, $z_n = \rho_n \sin\theta_n$, then $(y_n, z_n) \to (0,0)$ is equivalent to $\rho_n \to 0$.

$$\begin{aligned}
\lim_{(x_n,y_n,z_n,t_n)\to(0,0,0,0)} u(x_n,y_n,z_n,t_n) &= \lim_{(x_n,y_n,z_n,t_n)\to(0,0,0,0)} \frac{z_n(1-\sqrt{1-4(y_n^2+z_n^2)})}{2(y_n^2+z_n^2)} \\
&= \lim_{\rho_n\to 0} \frac{\rho_n \sin\theta_n (1-\sqrt{1-4\rho_n^2})}{2\rho_n^2} \\
&= \lim_{\rho_n\to 0} \frac{\sin\theta_n (1-\sqrt{1-4\rho_n^2})}{2\rho_n} \\
&= \lim_{\rho_n\to 0} \sin\theta_n (4\rho(1-4\rho_n^2)^{-1/2})) \qquad \text{(L'Hoppital's rule)} \\
&= 0 = u(0,0,0,0).
\end{aligned}$$

And in a similar fashion we get

$$\lim_{(x_n,y_n,z_n,t_n)\to(0,0,0,0)} v(x_n,y_n,z_n,t_n) = 0 = v(0,0,0,0).$$

If we have $(x,y,z,t) \in \psi_1(\mathbb{R}^2)$, such that $4x^2+t^2 = 1/4$, then $u^2+v^2 = 1$, which is, as we saw before, the same as $y^2+z^2 = 1/4$, so in this case

$$\frac{z(1-\sqrt{1-4(y^2+z^2)})}{2(y^2+z^2)} = \frac{z(1+\sqrt{1-4(y^2+z^2)})}{2(y^2+z^2)},$$

and

$$\frac{y(1-\sqrt{1-4(y^2+z^2)})}{2(y^2+z^2)} = \frac{y(1+\sqrt{1-4(y^2+z^2)})}{2(y^2+z^2)}.$$

So the expressions are equivalent in the intersection of their domains, then the whole expression giving (u,v) from (x,y,z,t) is continuous.

Using these expressions, we can see that $\psi^{-1}(x,y,z,t) = (u(x,y,z,t), v(x,y,z,t))$, and so ψ_1 is a continuous map, bijective onto its image, and its inverse is also continuous, then we conclude that $\psi_1 : \mathbb{R}^2 \to \psi_1(\mathbb{R}^2)$ is a homeomorphism. Homeomorphisms map open sets to open sets, then, as $\mathbb{R}^2$ is an open set, $\psi_1(\mathbb{R}^2)$ is an open subset of $\mathcal{H}(S^2)$.

Now consider ψ_2. It is possible to see that if $\psi_1(u,v) = (x_1,y_1,z_1,t_1)$ and $\psi_2(u,v) = (x_2,y_2,z_2,t_2)$ then $x_2 = z_1$ and $y_2 = y_1$, then the same thing that happens for ψ_1 related to y and z occurs to ψ_2 related to y and x, respectively. Using the result obtained for ψ_1, if $\psi_2(u, v) = (x, y, z, t)$, then

$$x^2 + y^2 \leq 1/4 \qquad \text{and } x^2 + y^2 = 1/4 \text{ iff } u^2 + v^2 = 1.$$

Also u and v are solutions of the equations

$$(x^2 + y^2)u^2 - xu + x^2 = 0$$

$$(x^2 + y^2)v^2 - yv + y^2 = 0.$$

And we have

$$u = \begin{cases} 0 & \text{if } x = y = 0; \\ \dfrac{x(1 - \sqrt{1 - 4(x^2 + y^2)})}{2(x^2 + y^2)} & \text{if } u^2 + v^2 \leq 1; \\ \dfrac{x(1 + \sqrt{1 - 4(x^2 + y^2)})}{2(x^2 + y^2)} & \text{if } u^2 + v^2 \geq 1. \end{cases}$$

$$v = \begin{cases} 0 & \text{if } x = y = 0; \\ \dfrac{y(1 - \sqrt{1 - 4(x^2 + y^2)})}{2(x^2 + y^2)} & \text{if } u^2 + v^2 \leq 1; \\ \dfrac{y(1 + \sqrt{1 - 4(x^2 + y^2)})}{2(x^2 + y^2)} & \text{if } u^2 + v^2 \geq 1. \end{cases}$$

Those expressions are continuous in their domains, and then ψ_2 is a continuous map, bijective onto its image, and its inverse is also continuous, then we conclude that $\psi_2 : \mathbb{R}^2 \to \psi_2(\mathbb{R}^2)$ is a homeomorphism, and $\psi_2(\mathbb{R}^2)$ is an open subset of $\mathcal{H}(S^2)$.

The same applies to ψ_3. It is possible to see that if $\psi_1(u, v) = (x_1, y_1, z_1, t_1)$ and $\psi_3(u, v) = (x_3, y_3, z_3, t_3)$ then $x_3 = z_1$ and $z_3 = y_1$, then the same thing that happens for ψ_1 related to y and z occurs to ψ_3 related to z and x, respectively. Using the result obtained for ψ_1, if $\psi_3(u, v) = (x, y, z, t)$, then

$$x^2 + z^2 \leq 1/4 \qquad \text{and } x^2 + z^2 = 1/4 \text{ iff } u^2 + v^2 = 1.$$

Also u and v are solutions of the equations

$$(x^2 + y^2)u^2 - xu + x^2 = 0$$

$$(x^2 + y^2)v^2 - zv + z^2 = 0.$$

And we have

$$u = \begin{cases} 0 & \text{if } x = z = 0; \\ \dfrac{x(1 - \sqrt{1 - 4(x^2 + z^2)})}{2(x^2 + z^2)} & \text{if } u^2 + v^2 \leq 1; \\ \dfrac{x(1 + \sqrt{1 - 4(x^2 + z^2)})}{2(x^2 + z^2)} & \text{if } u^2 + v^2 \geq 1. \end{cases}$$

$$v = \begin{cases} 0 & \text{if } x = z = 0; \\ \dfrac{z(1-\sqrt{1-4(x^2+z^2)})}{2(x^2+z^2)} & \text{if } u^2+v^2 \leq 1; \\ \dfrac{z(1+\sqrt{1-4(x^2+z^2)})}{2(x^2+z^2)} & \text{if } u^2+v^2 \geq 1. \end{cases}$$

Those expressions are continuous in their domains, and then ψ_3 is a continuous map, bijective onto its image, and its inverse is also continuous, then we conclude that $\psi_3 : \mathbb{R}^2 \to \psi_2(\mathbb{R}^2)$ is a homeomorphism, and $\psi_3(\mathbb{R}^2)$ is an open subset of $\mathcal{H}(S^2)$.

Now let's prove that the union of the U_i's covers $\mathcal{H}(S^2)$. First, we can see that the image of α_1 is the hemisphere $H_z = \{(x,y,z) \in S^2; z > 0\}$. In fact, on one hand, it is clear that $\alpha_1(\mathbb{R}^2) \subset H_z$; on the other hand, let $(x,y,z) \in H_z$, and make $u = x/z$ and $v = y/z$, then

$$\sqrt{u^2+v^2+1} = \sqrt{x^2/z^2+y^2/z^2+1} = \sqrt{(x^2+y^2+z^2)/z^2} = \sqrt{1/z^2} = 1/z,$$

$$\begin{aligned} \alpha_1(u,v) &= \alpha_1(x/z,y/z) \\ &= \left(\frac{u}{\sqrt{u^2+v^2+1}}, \frac{v}{\sqrt{u^2+v^2+1}}, \frac{1}{\sqrt{u^2+v^2+1}}\right) \\ &= \left(\frac{x/z}{1/z}, \frac{v}{1/z}, \frac{1}{1/z}\right) \\ &= (x,y,z). \end{aligned}$$

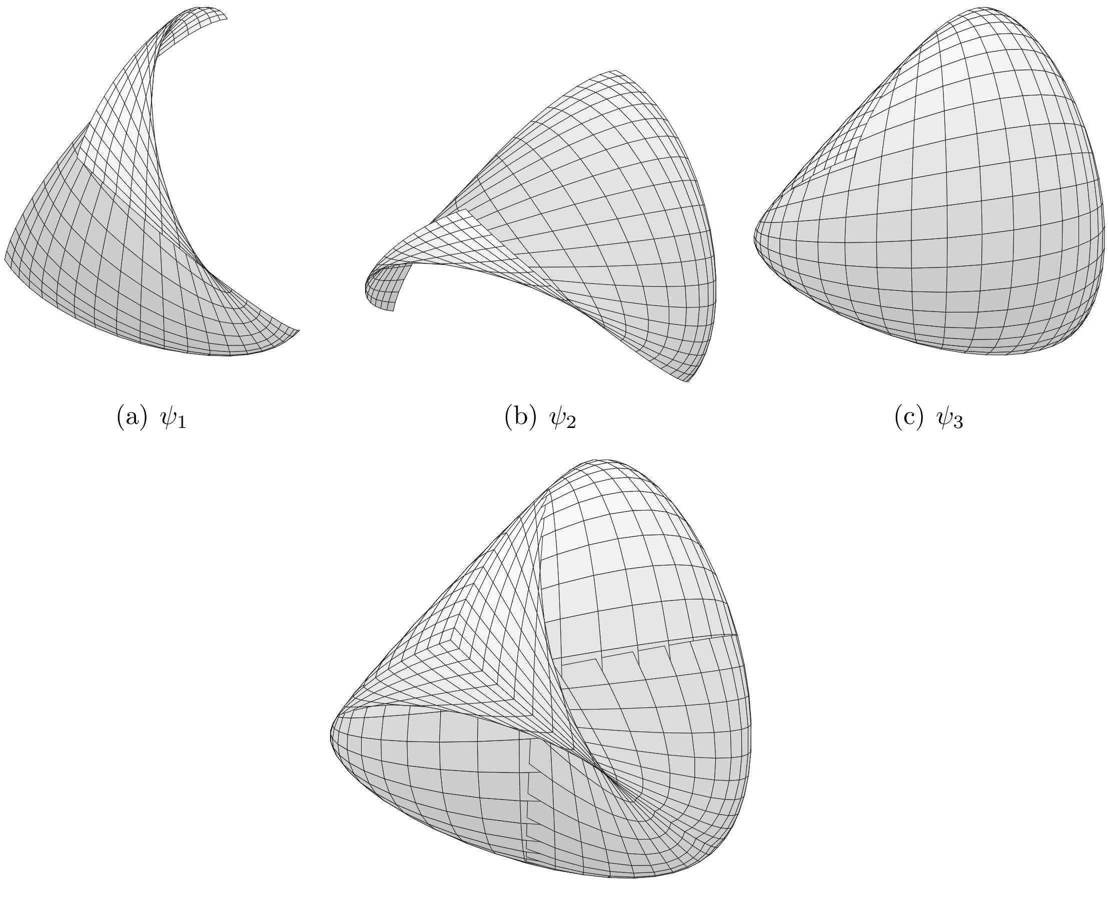
So $H_z \subset \alpha_1(\mathbb{R}^2)$. Then $H_z = \alpha_1(\mathbb{R}^2)$. Similarly the image of α_2 is the hemisphere $H_y = \{(x,y,z) \in S^2; y > 0\}$, and the image of α_3 is the hemisphere $H_x = \{(x,y,z) \in S^2; x > 0\}$.

Now let $p \in \mathcal{H}(S^2)$, so there is some $s = (x_s, y_s, z_s) \in S^2$ such that $\mathcal{H}(s) = p$. If $z_s > 0$ then $s \in H_z = \alpha_1(\mathbb{R}^2)$, so there is some $(u,v) \in \mathbb{R}^2$ such that $s = \alpha_1(u,v)$, and then $p = \mathcal{H}(\alpha_1(u,v)) = \psi_1(u,v)$, so $p \in \psi_1(\mathbb{R}^2) = U_1(\mathbb{R}^2)$. If $y_s > 0$ then $s \in H_y = \alpha_2(\mathbb{R}^2)$, so there is some $(u,v) \in \mathbb{R}^2$ such that $s = \alpha_2(u,v)$, and then $p = \mathcal{H}(\alpha_2(u,v)) = \psi_2(u,v)$, so $p \in \psi_2(\mathbb{R}^2) = U_2(\mathbb{R}^2)$. If $x_s > 0$ then $s \in H_x = \alpha_3(\mathbb{R}^2)$, so there is some $(u,v) \in \mathbb{R}^2$ such that $s = \alpha_3(u,v)$, and then $p = \mathcal{H}(\alpha_3(u,v)) = \psi_3(u,v)$, so $p \in \psi_3(\mathbb{R}^2) = U_3(\mathbb{R}^2)$. For s such that $x_s < 0$, $y_s < 0$ and $z_s < 0$, we can choose the point $s' = -s$, because $\mathcal{H}(s') = \mathcal{H}(s)$. As s belongs to S^2, $s \neq 0$, then at least one of its coordinates is non-zero, and if it is negative we can use $s' = -s$, so we will end up in some of the hemispheres H_x, H_y and H_z. So p belongs to $U_1 \cup U_2 \cup U_3$.

We conclude observing that for every point $p \in \mathcal{H}(S^2)$ there is an open set $U_i \subset \mathcal{H}(S^2)$, with $p \in U_i$ and a function $\psi_i : \mathbb{R}^2 \to \mathbb{R}^4$, such that ϕ_i is a homeomorphism between $\mathbb{R}^2$ and $U_i = \psi_i(\mathbb{R}^2)$, and $(d\psi_i)_{t0}$ is injective for $t_0 = \psi^{-1}(p)$ (as ψ_i is nonsingular). So ψ_1, ψ_2 and ψ_3 are parametrizations of a smooth manifold $\mathcal{H}(S^2)$.

d) In Figure 6 there are plots of each surface ψ_i separately, and they together, where the fourth coordinates were dropped.

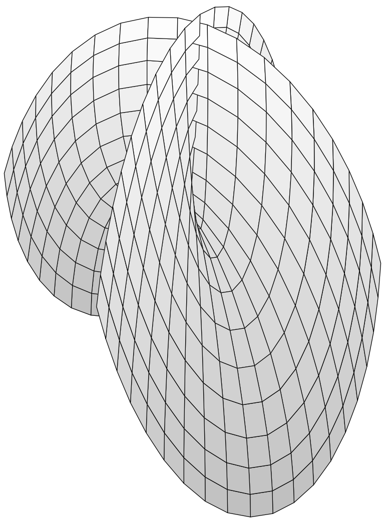
In Figure 7 there are plots of each surface ψ_i separately, and they together, where the third coordinates were dropped.



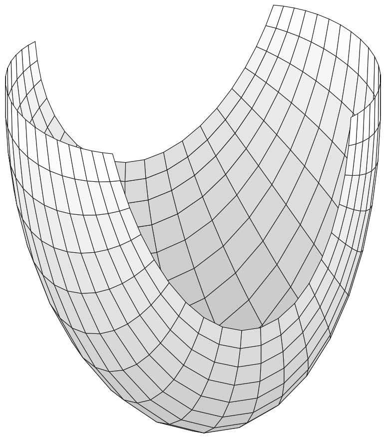
(a) ψ_1 (b) ψ_2 (c) ψ_3

(d) All the functions together

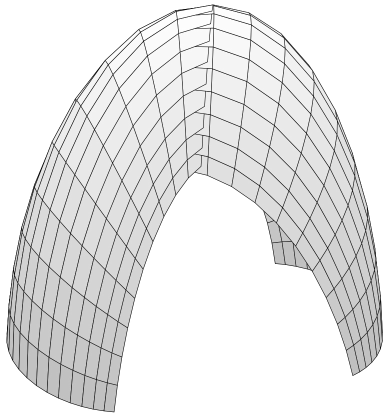
Figure 6: Surface obtained by dropping the fourth coordinate of the functions ψ_1, ψ_1 and ψ_3.



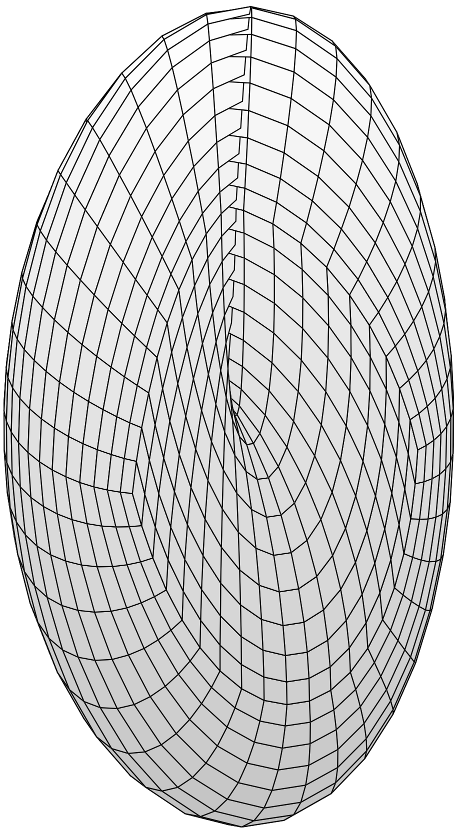
(a) ψ_1



(b) ψ_2



(c) ψ_3



(d) All the functions together

Figure 7: Surface obtained by dropping the third coordinate of the functions ψ_1, ψ_1 and ψ_3.

e) Let $(x,y,z,t) \in \mathcal{H}(S^2)$. We will prove that

$$\begin{aligned} x^2y^2 + x^2z^2 + y^2z^2 &= xyz \\ x(z^2 - y^2) &= yzt. \end{aligned}$$

As we saw, the union of the U_i's covers $\mathcal{H}(S^2)$, so there must be some $(u,v) \in \mathbb{R}^2$ and an $i \in \{1,2,3\}$, such that $\psi_i(u,v) = (x,y,z,t)$.

First suppose that $i = 1$, then $\psi_1(u,v) = (x,y,z,t)$. Observe that

$$\begin{aligned} x^2y^2 + x^2z^2 + y^2z^2 &= \left(\frac{uv}{u^2+v^2+1}\right)^2 \left(\frac{v}{u^2+v^2+1}\right)^2 \\ &\quad + \left(\frac{uv}{u^2+v^2+1}\right)^2 \left(\frac{u}{u^2+v^2+1}\right)^2 \\ &\quad + \left(\frac{v}{u^2+v^2+1}\right)^2 \left(\frac{u}{u^2+v^2+1}\right)^2 \\ &= \frac{(u^2v^2)v^2}{(u^2+v^2+1)^4} + \frac{(u^2v^2)u^2}{(u^2+v^2+1)^4} + \frac{u^2v^2}{(u^2+v^2+1)^4} \\ &= \frac{(u^2v^2)(v^2+u^2+1)}{(u^2+v^2+1)^4} \\ &= \frac{u^2v^2}{(u^2+v^2+1)^3}. \end{aligned}$$

And that

$$\begin{aligned} xyz &= \left(\frac{uv}{u^2+v^2+1}\right) \left(\frac{v}{u^2+v^2+1}\right) \left(\frac{u}{u^2+v^2+1}\right) \\ &= \frac{u^2v^2}{(u^2+v^2+1)^3}. \end{aligned}$$

Then $x^2y^2 + x^2z^2 + y^2z^2 = xyz$.

Now observe that

$$\begin{aligned} x(z^2 - y^2) &= \left(\frac{uv}{u^2+v^2+1}\right) \left(\left(\frac{u}{u^2+v^2+1}\right)^2 - \left(\frac{v}{u^2+v^2+1}\right)^2 \right) \\ &= \frac{uv(u^2-v^2)}{(u^2+v^2+1)^3}. \end{aligned}$$

And that

$$\begin{aligned} yzt &= \left(\frac{v}{u^2+v^2+1}\right) \left(\frac{u}{u^2+v^2+1}\right) \left(\frac{u^2-v^2}{u^2+v^2+1}\right) \\ &= \frac{uv(u^2-v^2)}{(u^2+v^2+1)^3}. \end{aligned}$$

Then $x(z^2 - y^2) = yzt$.

Suppose now that $i = 2$, then $\psi_2(u, v) = (x, y, z, t)$. Observe that

$$\begin{aligned} x^2y^2 + x^2z^2 + y^2z^2 &= \left(\frac{u}{u^2+v^2+1}\right)^2 \left(\frac{v}{u^2+v^2+1}\right)^2 \\ &\quad + \left(\frac{u}{u^2+v^2+1}\right)^2 \left(\frac{uv}{u^2+v^2+1}\right)^2 \\ &\quad + \left(\frac{v}{u^2+v^2+1}\right)^2 \left(\frac{uv}{u^2+v^2+1}\right)^2 \\ &= \frac{(u^2v^2)(1+u^2+v^2)}{(u^2+v^2+1)^4} \\ &= \frac{u^2v^2}{(u^2+v^2+1)^3}. \end{aligned}$$

And that

$$\begin{aligned} xyz &= \left(\frac{u}{u^2+v^2+1}\right) \left(\frac{v}{u^2+v^2+1}\right) \left(\frac{uv}{u^2+v^2+1}\right) \\ &= \frac{u^2v^2}{(u^2+v^2+1)^3}. \end{aligned}$$

Then $x^2y^2 + x^2z^2 + y^2z^2 = xyz$.
Now observe that

$$\begin{aligned} x(z^2-y^2) &= \left(\frac{u}{u^2+v^2+1}\right) \left(\left(\frac{uv}{u^2+v^2+1}\right)^2 - \left(\frac{v}{u^2+v^2+1}\right)^2 \right) \\ &= \frac{u(u^2v^2-v^2)}{(u^2+v^2+1)^3} \\ &= \frac{uv^2(u^2-1)}{(u^2+v^2+1)^3}. \end{aligned}$$

And that

$$\begin{aligned} yzt &= \left(\frac{v}{u^2+v^2+1}\right) \left(\frac{uv}{u^2+v^2+1}\right) \left(\frac{u^2-1}{u^2+v^2+1}\right) \\ &= \frac{uv^2(u^2-1)}{(u^2+v^2+1)^3}. \end{aligned}$$

Then $x(z^2 - y^2) = yzt$.

Finally, suppose that $i = 3$, then $\psi_3(u, v) = (x, y, z, t)$. Observe that

$$\begin{aligned} x^2y^2 + x^2z^2 + y^2z^2 &= \left(\frac{u}{u^2+v^2+1}\right)^2 \left(\frac{uv}{u^2+v^2+1}\right)^2 \\ &\quad + \left(\frac{u}{u^2+v^2+1}\right)^2 \left(\frac{v}{u^2+v^2+1}\right)^2 \\ &\quad + \left(\frac{uv}{u^2+v^2+1}\right)^2 \left(\frac{v}{u^2+v^2+1}\right)^2 \\ &= \frac{(u^2v^2)(u^2+v^2+1)}{(u^2+v^2+1)^4} \\ &= \frac{u^2v^2}{(u^2+v^2+1)^3}. \end{aligned}$$

And that

$$\begin{aligned} xyz &= \left(\frac{u}{u^2+v^2+1}\right)\left(\frac{uv}{u^2+v^2+1}\right)\left(\frac{v}{u^2+v^2+1}\right) \\ &= \frac{u^2v^2}{(u^2+v^2+1)^3}. \end{aligned}$$

Then $x^2y^2 + x^2z^2 + y^2z^2 = xyz$.
Now observe that

$$\begin{aligned} x(z^2-y^2) &= \left(\frac{u}{u^2+v^2+1}\right)\left(\left(\frac{v}{u^2+v^2+1}\right)^2 - \left(\frac{uv}{u^2+v^2+1}\right)^2\right) \\ &= \frac{u(v^2-u^2v^2)}{(u^2+v^2+1)^3} \\ &= \frac{uv^2(1-u^2)}{(u^2+v^2+1)^3}. \end{aligned}$$

And that

$$\begin{aligned} yzt &= \left(\frac{uv}{u^2+v^2+1}\right)\left(\frac{v}{u^2+v^2+1}\right)\left(\frac{1-u^2}{u^2+v^2+1}\right) \\ &= \frac{uv^2(1-u^2)}{(u^2+v^2+1)^3}. \end{aligned}$$

Then $x(z^2-y^2) = yzt$.

Consider the point $p = (x_p, y_p, z_p, t_p) = (0, 0, 0, 0.5)$.

Using the inverse of ψ_1, we see that the only point (x, y, z, t) in $\psi_1(\mathbb{R}^2)$ such that $y = y_p = 0$ and $z = z_p = 0$ is $\psi_1(0, 0) = (0, 0, 0, 0) \neq p$. So we conclude that $p \notin \psi_1(\mathbb{R}^2)$.

Using the inverse of ψ_2, we see that the only point (x, y, z, t) in $\psi_2(\mathbb{R}^2)$ such that $x = x_p = 0$ and $y = y_p = 0$ is $\psi_2(0, 0) = (0, 0, 0, -1) \neq p$. So we conclude that $p \notin \psi_2(\mathbb{R}^2)$.

Using the inverse of ψ_3, we see that the only point (x, y, z, t) in $\psi_3(\mathbb{R}^2)$ such that $x = x_p = 0$ and $z = z_p = 0$ is $\psi_3(0, 0) = (0, 0, 0, 1) \neq p$. So we conclude that $p \notin \psi_3(\mathbb{R}^2)$.

As the union of U_i's covers $\mathcal{H}(S^2$ then $p \notin \mathcal{H}(S^2)$. But, observe that

$$x_p^2y_p^2 + x_p^2z_p^2 + y_p^2z_p^2 = 0 = x_py_pz_p,$$

and

$$x_p(z_p^2 - y_p^2) = 0 = y_pz_pt_p,$$

so p is in the zero locus of these equations, but $p \notin \mathcal{H}(S^2)$, this means that the zero locus of these equations strictly contains $\mathcal{H}(S^2)$.