Computational Manifolds and Applications–2011, IMPA

Homework 3

Due October 25, 2011

Problem 1. Consider the parametric surface given by

$$
\begin{aligned}
x(u,v) &= \frac{4v(u^2+v^2-1)}{(u^2+v^2+1)^2},\\
y(u,v) &= \frac{4u(u^2+v^2-1)}{(u^2+v^2+1)^2},\\
z(u,v) &= \frac{4(u^2-v^2)}{(u^2+v^2+1)^2}.
\end{aligned}
$$

The trace of this surface is called the *Steiner Roman surface*. In order to plot this surface, make the change of variables

$$
\begin{aligned}
u &= \rho\cos\theta\\
v &= \rho\sin\theta.
\end{aligned}
$$

Prove that we obtain the parametric definition

$$
\begin{aligned}
x &= \frac{4\rho(\rho^2-1)}{(\rho^2+1)^2}\,\sin\theta,\\
y &= \frac{4\rho(\rho^2-1)}{(\rho^2+1)^2}\,\cos\theta,\\
z &= \frac{4\rho^2}{(\rho^2+1)^2}\,\cos 2\theta.
\end{aligned}
$$

Show that the entire trace of the surface is obtained for $\rho \in [0,1]$ and $\theta \in [-\pi,\pi]$. Plot the trace of the surface using the above parametrization.

Plot the portion of the surface for $\rho \in [0,1]$ and $\theta \in [0,\pi]$.

Prove that this surface has four singular points.

(b) Express the trigonometric functions in terms of $u = \tan(\theta/2)$, and letting $v = \rho$, show that we get

$$\begin{aligned} x &= \frac{8uv(u^2+1)(v^2-1)}{(u^2+1)^2(v^2+1)^2}, \\ y &= \frac{4v(1-u^4)(v^2-1)}{(u^2+1)^2(v^2+1)^2}, \\ z &= \frac{4v^2(u^4-6u^2+1)}{(u^2+1)^2(v^2+1)^2}. \end{aligned}$$

Problem 2. Consider the parametric surface given by

$$\begin{aligned} x &= \frac{4u(1-u^2)((a+r)v^4+2(a-3r)v^2+a+r)}{(u^2+1)^2(v^2+1)^2}, \\ y &= \frac{4rv(1-u^4)(1-v^2)}{(u^2+1)^2(v^2+1)^2}, \\ z &= \frac{8ruv(1+u^2)(1-v^2)}{(u^2+1)^2(v^2+1)^2}, \end{aligned}$$

where $a > r > 1$.

Show that the entire surface is obtained for $u, v \in [-1, 1]$, and plot this surface for $a = 2$ and $r = 1$.

The above surface is a projection of the Klein bottle in $\mathbb{R}^4$ into $\mathbb{R}^3$. Can you see why?

Problem 3. Consider the parametric surface given by

$$\begin{aligned} x &= \frac{(u^4-6u^2+1)((a+r)v^4+2(a-3r)v^2+a+r)}{(u^2+1)^2(v^2+1)^2}, \\ y &= \frac{4u(1-u^2)((a+r)v^4+2(a-3r)v^2+a+r)}{(u^2+1)^2(v^2+1)^2}, \\ z &= \frac{8ruv(1+u^2)(1-v^2)}{(u^2+1)^2(v^2+1)^2}, \end{aligned}$$

where $a > r > 1$.

Show that the entire surface is obtained for $u, v \in [-1, 1]$, and plot this surface for $a = 2$ and $r = 1$.

Plot the portion of the surface for $u \in [-1, 1]$ and $v \in [0, 1]$.

The above surface is another projection of the Klein bottle in $\mathbb{R}^4$ into $\mathbb{R}^3$. Can you see why?

Problem 4. (a) Consider the map $\mathcal{H}\colon \mathbb{R}^3 \to \mathbb{R}^4$ defined such that

$$(x, y, z) \mapsto (xy, yz, xz, x^2 - y^2).$$

Prove that when it is restricted to the sphere S^2 (in $\mathbb{R}^3$), we have $\mathcal{H}(x, y, z) = \mathcal{H}(x', y', z')$ iff $(x', y', z') = (x, y, z)$ or $(x', y', z') = (-x, -y, -z)$. In other words, the inverse image of every point in $\mathcal{H}(S^2)$ consists of two antipodal points.

Prove that the map $\mathcal{H}$ induces an injective map from the projective plane onto $\mathcal{H}(S^2)$, and that it is a homeomorphism.

(b) The map $\mathcal{H}$ allows us to realize concretely the projective plane in $\mathbb{R}^4$ as an embedded manifold. Consider the three maps from $\mathbb{R}^2$ to $\mathbb{R}^4$ given by

$$\begin{aligned}
\psi_1(u, v) &= \left(\frac{uv}{u^2 + v^2 + 1}, \frac{v}{u^2 + v^2 + 1}, \frac{u}{u^2 + v^2 + 1}, \frac{u^2 - v^2}{u^2 + v^2 + 1}\right), \\
\psi_2(u, v) &= \left(\frac{u}{u^2 + v^2 + 1}, \frac{v}{u^2 + v^2 + 1}, \frac{uv}{u^2 + v^2 + 1}, \frac{u^2 - 1}{u^2 + v^2 + 1}\right), \\
\psi_3(u, v) &= \left(\frac{u}{u^2 + v^2 + 1}, \frac{uv}{u^2 + v^2 + 1}, \frac{v}{u^2 + v^2 + 1}, \frac{1 - u^2}{u^2 + v^2 + 1}\right).
\end{aligned}$$

Observe that ψ_1 is the composition $\mathcal{H} \circ \alpha_1$, where $\alpha_1\colon \mathbb{R}^2 \longrightarrow S^2$ is given by

$$(u, v) \mapsto \left(\frac{u}{\sqrt{u^2 + v^2 + 1}}, \frac{v}{\sqrt{u^2 + v^2 + 1}}, \frac{1}{\sqrt{u^2 + v^2 + 1}}\right),$$

that ψ_2 is the composition $\mathcal{H} \circ \alpha_2$, where $\alpha_2\colon \mathbb{R}^2 \longrightarrow S^2$ is given by

$$(u, v) \mapsto \left(\frac{u}{\sqrt{u^2 + v^2 + 1}}, \frac{1}{\sqrt{u^2 + v^2 + 1}}, \frac{v}{\sqrt{u^2 + v^2 + 1}}\right).$$

and ψ_3 is the composition $\mathcal{H} \circ \alpha_3$, where $\alpha_3\colon \mathbb{R}^2 \longrightarrow S^2$ is given by

$$(u, v) \mapsto \left(\frac{1}{\sqrt{u^2 + v^2 + 1}}, \frac{u}{\sqrt{u^2 + v^2 + 1}}, \frac{v}{\sqrt{u^2 + v^2 + 1}}\right),$$

Prove that each ψ_i is injective, continuous and nonsingular (i.e., the Jacobian is never zero).

(c) Prove that if $\psi_1(u, v) = (x, y, z, t)$, then

$$y^2 + z^2 \leq \frac{1}{4} \quad \text{and} \quad y^2 + z^2 = \frac{1}{4} \quad \text{iff} \quad u^2 + v^2 = 1.$$

Prove that u and v are solutions of the equations

$$\begin{aligned}
(y^2 + z^2)u^2 - zu + z^2 &= 0 \\
(y^2 + z^2)v^2 - yv + y^2 &= 0.
\end{aligned}$$

Prove that if $y^2 + z^2 \neq 0$, then

$$u = \frac{z(1 - \sqrt{1 - 4(y^2 + z^2)})}{2(y^2 + z^2)} \quad \text{if} \quad u^2 + v^2 \leq 1,$$

else

$$u = \frac{z(1 + \sqrt{1 - 4(y^2 + z^2)})}{2(y^2 + z^2)} \quad \text{if} \quad u^2 + v^2 \geq 1,$$

and there are similar formulae for v. Prove that the expression giving u in terms of y and z is continuous everywhere in $\{(y, z) \mid y^2 + z^2 \leq \frac{1}{4}\}$ and similarly for the expression giving v in terms of y and z. Conclude that $\psi_1 \colon \mathbb{R}^2 \to \psi_1(\mathbb{R}^2)$ is a homeomorphism onto its image. Therefore, $U_1 = \psi_1(\mathbb{R}^2)$ is an open subset of $\mathcal{H}(S^2)$.

Prove that if $\psi_2(u, v) = (x, y, z, t)$, then

$$x^2 + y^2 \leq \frac{1}{4} \quad \text{and} \quad x^2 + y^2 = \frac{1}{4} \quad \text{iff} \quad u^2 + v^2 = 1.$$

Prove that u and v are solutions of the equations

$$(x^2 + y^2)u^2 - xu + x^2 = 0$$
$$(x^2 + y^2)v^2 - yv + y^2 = 0.$$

Conclude that $\psi_2 \colon \mathbb{R}^2 \to \psi_2(\mathbb{R}^2)$ is a homeomorphism onto its image and that the set $U_2 = \psi_2(\mathbb{R}^2)$ is an open subset of $\mathcal{H}(S^2)$.

Prove that if $\psi_3(u, v) = (x, y, z, t)$, then

$$x^2 + z^2 \leq \frac{1}{4} \quad \text{and} \quad x^2 + z^2 = \frac{1}{4} \quad \text{iff} \quad u^2 + v^2 = 1.$$

Prove that u and v are solutions of the equations

$$(x^2 + z^2)u^2 - xu + x^2 = 0$$
$$(x^2 + z^2)v^2 - zv + z^2 = 0.$$

Conclude that $\psi_3 \colon \mathbb{R}^2 \to \psi_3(\mathbb{R}^2)$ is a homeomorphism onto its image and that the set $U_3 = \psi_3(\mathbb{R}^2)$ is an open subset of $\mathcal{H}(S^2)$.

Prove that the union of the U_i's covers $\mathcal{H}(S^2)$. Conclude that ψ_1, ψ_2, ψ_3 are parametrizations of $\mathbb{RP}^2$ as a smooth manifold in $\mathbb{R}^4$.

(d) Plot the surfaces obtained by dropping the fourth coordinate and the third coordinates, respectively (with $u, v \in [-1, 1]$).

(e) Prove that if $(x, y, z, t) \in \mathcal{H}(S^2)$, then

$$\begin{aligned} x^2y^2 + x^2z^2 + y^2z^2 &= xyz \\ x(z^2 - y^2) &= yzt. \end{aligned}$$

Prove that the zero locus of these equations strictly contains $\mathcal{H}(S^2)$. This is a "famous mistake" of Hilbert and Cohn-Vossen in *Geometry and the Immagination*!

Finding a set of equations defining exactly $\mathcal{H}(S^2)$ appears to be an open problem.