Computational Manifolds and Applications - 2011

Leonardo de Oliveira Carvalho

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1 Problem 1

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function given by

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^4+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Let's compute the directional derivative $D_u f(0,0)$ of f at $(0,0)$ for every vector $u = (u_1, u_2) \neq 0$.

$$\begin{aligned} D_u f(0,0) &= \lim_{t\to 0} \frac{f((0,0)+t(u_1,u_2)) - f(0,0)}{t} \\ &= \lim_{t\to 0} \frac{f(tu_1, tu_2)}{t} \\ &= \lim_{t\to 0} \frac{(tu_1)^2(tu_2)}{(tu_1)^4+(tu_2)^2} \cdot \frac{1}{t} \\ &= \lim_{t\to 0} \frac{t^3 u_1^2 u_2}{t^4 u_1^4 + t^2 u_2^2} \cdot \frac{1}{t} \\ &= \lim_{t\to 0} \frac{u_1^2 u_2}{t^2 u_1^4 + u_2^2} \\ &= \frac{u_1^2 u_2}{0^2 u_1^4 + u_2^2} \\ &= \frac{u_1^2 u_2}{u_2^2} \\ &= \frac{u_1^2}{u_2}. \end{aligned}$$

This is only valid when $u_2 \neq 0$. If $u_1 \neq 0, u_2 = 0$, then

$$\begin{aligned} D_u f(0,0) &= \lim_{t\to 0} \frac{f((0,0)+t(u_1,0)) - f(0,0)}{t} \\ &= \lim_{t\to 0} \frac{f(tu_1, 0)}{t} \\ &= \lim_{t\to 0} \frac{(tu_1)^2 \cdot 0}{(tu_1)^4 + 0^2} \cdot \frac{1}{t} \\ &= \lim_{t\to 0} 0 \\ &= 0 \end{aligned}$$

(b) We can prove that the derivative $Df(0,0)$ does not exist. Suppose that it does exist, so for every $u \neq 0$ we can calculate the derivative using a directional derivative using the relation $Df(0,0)(u) = D_u f(0,0)$. Using the results obtained above we observe that

$$Df(0,0)(1,0) = D_{(1,0)}f(0,0) = 0$$
$$Df(0,0)(0,1) = D_{(0,1)}f(0,0) = 0.$$

These implies that $Df(0,0) = 0$, because it is a linear application. But if we use $u = (1,1)$:

$$Df(0,0)(1,1) = D_{(1,1)}f(0,0) = 1.$$

So we have a contradiction, then the derivative does not exist.

In the parabola $y = x^2$, we have

$$f(x,y) = \begin{cases} \frac{x^2(x^2)}{x^4+x^2} = \frac{x^4}{2x^4} = 1/2 & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

So the function is constant, with a discontinuity at $(x,y) = (0,0)$.

2 Problem 2

(a) Let $f : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be the function defined on $n \times n$ matrices by

$$f(A) = A^2.$$

Let's calculate $Df_A(H)$, for $A, H \in M_N(\mathbb{R})$. Observe that

$$\begin{aligned} f(A+H) &= (A+H)(A+H) \\ &= A^2 + AH + HA + H^2 \\ &= f(A) + (AH + HA) + H^2. \end{aligned}$$

Define $L(H) = AH + HA$, and $\epsilon(H) = H^2/\|H\|$. We see that

$$\begin{aligned} \lim_{H\to 0} \|\epsilon(H)\| &= \lim_{H\to 0} \|H^2/\|H\|\| \\ &= \lim_{H\to 0} \|H^2\|/\|H\| \\ &\leq \lim_{H\to 0} \|H\|^2/\|H\| \\ &= \lim_{H\to 0} \|H\| \\ &= 0. \end{aligned}$$

So

$$\lim_{H\to 0} \epsilon(H) = 0.$$

Let $H_1, H_2 \in M_n(\mathbb{R}), \alpha \in \mathbb{R}$:

$$\begin{aligned} L(H_1 + H_2) &= A(H_1 + H_2) + (H_1 + H_2)A \\ &= AH_1 + AH_2 + H_1A + H_2A \\ &= AH_1 + H_1A + AH_2 + H_2A \\ &= L(H_1) + L(H_2), \end{aligned}$$

$$\begin{aligned} L(\alpha H_1) &= A(\alpha H_1) + (\alpha H_1)A \\ &= \alpha(AH_1 + H_1A) \\ &= \alpha L(H_1), \end{aligned}$$

so the function L is linear.

Then we can write

$$f(A+H) = f(A) + L(H) + \epsilon(H)\|H\|.$$

This proves that f is differentiable at A, and $Df(A) = L(H) = AH + HA$.

(b) Let $f : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ be the function defined on $n \times n$ matrices by

$$f(A) = A^3.$$

Let's calculate $Df_A(H)$, for $A, H \in M_N(\mathbb{R})$. Observe that

$$\begin{aligned} f(A+H) &= (A+H)(A+H)(A+H) \\ &= (A^2 + AH + HA + H^2)(A+H) \\ &= A^3 + AHA + HA^2 + H^2A + A^2H + AH^2 + HAH + H^3 \\ &= f(A) + (AHA + HA^2 + A^2H) + H^2A + AH^2 + HAH + H^3 \end{aligned}$$

Define $L(H) = AHA + HA^2 + A^2H$, and $\epsilon(H) = (H^2A + AH^2 + HAH + H^3)/\|H\|$. We see that

$$\begin{aligned} \lim_{H\to 0} \|\epsilon(H)\| &= \lim_{H\to 0} \|(H^2A + AH^2 + HAH + H^3)/\|H\|\| \\ &\le \lim_{H\to 0} (\|H^2A\| + \|AH^2\| + \|HAH\| + \|H^3\|)/\|H\| \\ &\le \lim_{H\to 0} (\|H\|^2\|A\| + \|A\|\|H\|^2 + \|H\|\|A\|\|H\| + \|H\|^3)/\|H\| \\ &\le \lim_{H\to 0} (\|H\|\|A\| + \|A\|\|H\| + \|A\|\|H\| + \|H\|^2) \\ &= 0. \end{aligned}$$

So

$$\lim_{H\to 0} \epsilon(H) = 0.$$

Let $H_1, H_2 \in M_n(\mathbb{R}), \alpha \in \mathbb{R}$:

$$\begin{aligned} L(H_1 + H_2) &= A(H_1 + H_2)A + (H_1 + H_2)A^2 + A^2(H_1 + H_2) \\ &= (AH_1 + AH_2)A + H_1A^2 + H_2A^2 + A^2H_1 + A^2H_2 \\ &= AH_1A + AH_2A + H_1A^2 + H_2A^2 + A^2H_1 + A^2H_2 \\ &= (AH_1A + H_1A^2 + A^2H_1) + (AH_2A + H_2A^2 + A^2H_2) \\ &= L(H_1) + L(H_2), \end{aligned}$$

$$\begin{aligned} L(\alpha H_1) &= A(\alpha H_1)A + (\alpha H_1)A^2 + A^2(\alpha H_1) \\ &= A(\alpha H_1)A + (\alpha H_1)A^2 + A^2(\alpha H_1) \\ &= \alpha(AH_1A + H_1A^2 + A^2H_1) \\ &= \alpha L(H_1), \end{aligned}$$

so the function L is linear.

Then we can write

$$f(A+H) = f(A) + L(H) + \epsilon(H)\|H\|.$$

This proves that f is differentiable at A, and $Df(A) = L(H) = AHA + HA^2 + A^2H$.

3 Problem 3

Let $F : GL(n, \mathbb{R}) \to M_n(\mathbb{R})$ be the function defined on invertible $n \times n$ matrices by

$$f(A) = A^{-1}.$$

We will prove that

$$Df_A(H) = -A^{-1}HA^{-1},$$

for all $A \in GL(n, \mathbb{R})$ and for all $H \in M_n(\mathbb{R})$.

For $H \in M_n(\mathbb{R})$ sufficiently small (in some matrix norm), such that $A+H$ is still invertible (which is possible because $GL(n, \mathbb{R})$ is an open set in $M_n(\mathbb{R})$), we have

$$\begin{aligned}
(A+H)^{-1} &= (A + AA^{-1}H)^{-1} \\
&= (A(I + A^{-1}H))^{-1} \\
&= (I + A^{-1}H)^{-1}A^{-1} \\
&= (I + A^{-1}H)^{-1}(I + A^{-1}H - A^{-1}H)A^{-1} \\
&= (I - (I + A^{-1}H)^{-1}(A^{-1}H))A^{-1} \\
&= A^{-1} - (I + A^{-1}H)^{-1}A^{-1}HA^{-1} \\
&= A^{-1} - ((I + A^{-1}H)^{-1}(I + A^{-1}H - A^{-1}H))A^{-1}HA^{-1} \\
&= A^{-1} - (I - (I + A^{-1}H)^{-1}A^{-1}H)A^{-1}HA^{-1} \\
&= A^{-1} - A^{-1}HA^{-1} + (I + A^{-1}H)^{-1}A^{-1}HA^{-1}HA^{-1} \\
&= A^{-1} - A^{-1}HA^{-1} + (A+H)^{-1}HA^{-1}HA^{-1}
\end{aligned}$$

Define $L(H) = -A^{-1}HA^{-1}$, $\epsilon(H) = ((A+H)^{-1}HA^{-1}HA^{-1})\|H\|$. Observe that

$$\begin{aligned}
\lim_{H\to 0} \|\epsilon(H)\| &= \lim_{H\to 0} \frac{\|(A+H)^{-1}HA^{-1}HA^{-1}\|}{\|H\|} \\
&\leq \lim_{H\to 0} \frac{\|(A+H)^{-1}\| \cdot \|H\| \cdot \|A^{-1}\| \cdot \|H\| \cdot \|A^{-1}\|}{\|H\|} \\
&= \lim_{H\to 0} \|(A+H)^{-1}\| \cdot \|A^{-1}\| \cdot \|H\| \cdot \|A^{-1}\| \\
&= \|A^{-1}\| \cdot \|A^{-1}\| \cdot \|0\| \cdot \|A^{-1}\| \\
&= 0.
\end{aligned}$$

So $\lim_{H\to 0} \epsilon(H) = 0$.

Let $H_1, H_2 \in M_n(\mathbb{R}), \alpha \in \mathbb{R}$:

$$\begin{aligned}
L(H_1 + H_2) &= -A(H_1 + H_2)A^{-1} \\
&= -AH_1A^{-1} - A^{-1}H_2A^{-1} \\
&= L(H_1) + L(H_2),
\end{aligned}$$

$$\begin{aligned} L(\alpha H_1) &= A^{-1}(\alpha H_1)A^{-1} \\ &= \alpha(A^{-1}(H_1)A^{-1}) \\ &= \alpha L(H_1), \end{aligned}$$

so the function L is linear.

Then we can write

$$f(A+H) = f(A) + L(H) + \epsilon(H)\|H\|.$$

This proves that f is differentiable at A, and $Df(A) = L(H) = -A^{-1}HA^{-1}$.

4 Problem 4

Let $\mathfrak{so}(n) = \{B \in M_n(\mathbb{R}); B^T = -B)\}$ be the set of skew-symmetric matrices.

(a) Let's check that the set $\mathfrak{so}(n)$ is a vector space of dimension $n(n-1)/2$.

Observe that if $B \in \mathfrak{so}(n)$ then $b_{ij} = -b_{ji}$, so $b_{ii} = -b_{ii}$ which implies that $b_{ii} = 0$, i.e. that all elements in the diagonal of B is equal to 0. We can define B using only the elements above the diagonal, i.e. the elements b_{ij} where $i < j$, because we can know the other elements using $b_{ji} = -b_{ij}$. In the first line, there are $n-1$ elements $b_{ij}, i < j$, (all but the diagonal element). In the second line, there are $n-2$ elements $b_{ij}, i < j$, (all but the diagonal and b_{21}). In the k^{th} line, there are $n-k$ elements $b_{ij}, i < j$, (all but the k elements $b_{kj}, j \leq k$). In the last line, there are no elements $b_{ij}, i < j$. So to define B we need $(n-1) + (n-2) + \cdots + 2 + 1 = n(n-1)/2$ elements (it is the sum of $n-1$ terms of a aritmetic progression with initial term 1 and common difference 1).

To see that this set is a vector space, we will use $B_1, B_2, B_3 \in \mathfrak{so}(n)$, $\alpha \in \mathbb{R}$. It is easy to see that the sum of B_1 and B_2 is still an element of $\mathfrak{so}(n)$, in fact for each i, j we have $b_{1ji} + b_{2ji} = -b_{1ij} - b_{2ij} = -(b_{1ij} + b_{2ij})$, so $(B_1 + B_2)^T = -(B_1 + B_2)$. Similarly αB_1 is also in $\mathfrak{so}(n)$, because for each i, j, $\alpha b_{1ji} = \alpha(-b_{1ij}) = -\alpha b_{1ij}$, so $(\alpha B)^T = -(\alpha B)$.

Let's check the necessary properties to $\mathfrak{so}(n)$ be a vector space.

Commutativity: Using the commutativity from the $M_n(\mathbb{R})$, we have $B_1 + B_2 = B_2 + B_1$, and we already know that the sum of elements of $\mathfrak{so}(n)$ is also an element of $\mathfrak{so}(n)$.

Associativity: Using the associativity from the $M_n(\mathbb{R})$, we have $(B_1 + B_2) + B_3 = B_1 + (B_2 + B_3)$.

Identity element in addition: The null matrix 0 in $M_n(\mathbb{R})$ also belongs to $\mathfrak{so}(n)$, because it is clear that $0^T = -0$. 0 is the identity element in addition in $M_n(\mathbb{R})$, so it is also the identity element in addition in $\mathfrak{so}(n)$.

Inverse elements in addition: If $B \in \mathfrak{so}(n)$, the matrix $-B$ is also in $\mathfrak{so}(n)$, and it is the inverse element of B, because $B + (-B) = 0$.

Distributivity: Given $\alpha, \beta \in \mathbb{R}$, and $B_1, B_2 \in \mathfrak{so}(n)$, we know from the distributivity of $M_n(\mathbb{R})$ that

$$(\alpha + \beta)B_1 = \alpha B_1 + \beta B_1,$$

and we know that $(\alpha + \beta)B_1, \alpha B_1, \beta B_1 \in \mathfrak{so}(n)$, then the sum is also in $\mathfrak{so}(n)$.

We also know from the distributivity of $M_n(\mathbb{R})$ that

$$\alpha(B_1 + B_2) = \alpha B_1 + \alpha B_2,$$

and we know that $\alpha(B_1 + B_2), \alpha B_1, \alpha B_2 \in \mathfrak{so}(n)$, then the sum is also in $\mathfrak{so}(n)$.

Identity element of scalar multiplication Given $B \in \mathfrak{so}(n)$, using the identity element of scalar multiplication of $M_n(\mathbb{R})$ we know that $1 \cdot B = B$, and we know that $1 \cdot B \in \mathfrak{so}(n)$, so 1 is also the identity element of scalar multiplication in $\mathfrak{so}(n)$.

Then we saw that $\mathfrak{so}(n)$ is a vector space of dimension $n(n-1)/2$, and thus is isomorphic to $\mathbb{R}^{n(n-1)/2}$.

Given an eigenvalue λ associated with the eigenvector v in $\mathbb{R}^n$ of a skew-symmetric matrix B, we have $Bv = \lambda v$, $\|v\| = 0$. Observe that

$$\lambda\langle v, v\rangle = \langle \lambda v, v\rangle = \langle Bv, v\rangle = \langle v, B^T v\rangle = \langle v, -Bv\rangle = \langle v, -\lambda v\rangle = \overline{(-\lambda)}\langle v, v\rangle,$$

so $\lambda = \overline{(-\lambda)}$, if we have $\lambda = a + bi$, then

$$a + bi = \overline{(-(a + bi))} = \overline{(-a - bi)} = -a + bi,$$

So $a = -a$, then $a = 0$, wich means that $\lambda = bi$, so λ is a pure imaginary number or 0.

Let's prove that $(I - B)$ is invertible. Let $v \in \mathbb{R}^n$, such that $(I - B)v = 0$, so we have $v - Bv = 0$, and $v = Bv$. Let's calculate the norm of v:

$$\|v\|^2 = \langle v, v\rangle = \langle Bv, v\rangle = \langle v, B^T v\rangle = \langle v, -Bv\rangle = \langle v, -v\rangle = -\langle v, v\rangle = -\|v\|^2,$$

so $\|v\| = 0$, then $v = 0$. So the only solution of $(I - B)v = 0$ is $v = 0$, this is a necessary and sufficient condition for $I - B$ be invertible.

Using this we can prove that $I + B$ is also invertible. We know that the transpose of an invertible matrix is also invertible, so $(I - B)^T$ is invertible, but we see that

$$(I - B)^T = (I^T - B^T) = (I + B),$$

so $I + B$ is invertible.

Let's now prove that $(I + B)(I - B) = (I - B)(I + B)$. We have

$$\begin{aligned}(I + B)(I - B) &= (I + B)I - (I + B)B \\ &= I + B - B - B^2 \\ &= I - B^2,\end{aligned}$$

and

$$\begin{aligned}(I - B)(I + B) &= (I - B)I + (I - B)B \\ &= I - B + B - B^2 \\ &= I - B^2 \\ &= (I + B)(I - B).\end{aligned}$$

Let's now prove that $(I + B)(I - B)^{-1} = (I - B)^{-1}(I + B)$. Using the result obtained above we have

$$\begin{aligned}
(I-B)(I+B) &= (I+B)(I-B) \\
(I-B)^{-1}(I-B)(I+B) &= (I-B)^{-1}(I+B)(I-B) \\
(I+B) &= (I-B)^{-1}(I+B)(I-B) \\
(I+B)(I-B)^{-1} &= (I-B)^{-1}(I+B)(I-B)(I-B)^{-1} \\
(I+B)(I-B)^{-1} &= (I-B)^{-1}(I+B).
\end{aligned}$$

Let $C : \mathfrak{so}(n) \to M_n(\mathbb{R})$ be the function given by

$$C(B) = (I-B)(I+B)^{-1}.$$

Let's show that $(C(B))^T C(B) = I$. Using the definition of C we have

$$\begin{aligned}
(C(B))^T C(B) &= ((I-B)(I+B)^{-1})^T (I-B)(I+B)^{-1} \\
&= ((I+B)^{-1})^T (I-B)^T (I-B)(I+B)^{-1} \\
&= ((I+B)^T)^{-1} (I-B^T)(I-B)(I+B)^{-1} \\
&= (I+B^T)^{-1}(I+B)(I-B)(I+B)^{-1} \\
&= (I-B)^{-1}(I+B)(I-B)(I+B)^{-1} \\
&= (I+B)(I-B)^{-1}(I-B)(I+B)^{-1} \\
&= (I+B)(I+B)^{-1} \\
&= I
\end{aligned}$$

Let's show that $\det C(B) = 1$:

$$\begin{aligned}
\det C(B) &= \det((I-B)(I+B)^{-1}) \\
&= \det(I-B)\det((I+B)^{-1}) \\
&= \frac{\det(I-B)}{\det(I+B)} \\
&= \frac{\det(I^T+B^T)}{\det(I+B)} \\
&= \frac{\det(I+B)^T}{\det(I+B)} \\
&= \frac{\det(I+B)}{\det(I+B)} \\
&= 1.
\end{aligned}$$

So we have $(C(B))^T C(B) = I$ and $\det C(B) = 1$, which means that $C(B)$ is a rotation matrix.

Suppose that $C(B)$ admits -1 as an eigenvalue, so there is $v \in \mathbb{R}^n$, $v \neq 0$ such that:

$$\begin{aligned}
C(B)v &= -v \\
(I-B)(I+B)^{-1}v &= -v \\
(I+(-B))(I-(-B))^{-1}v &= -v \\
(I-(-B))^{-1}(I+(-B))v &= -v \\
(I+B)^{-1}(I-B)v &= -v \\
(I+B)(I+B)^{-1}(I-B)v &= (I+B)(-v) \\
(I-B)v &= (I+B)(-v) \\
v - Bv &= -v - Bv \\
v &= -v \\
2v &= 0 \\
v &= 0,
\end{aligned}$$

this is a contradiction, so $C(B)$ does not admit -1 as an eigenvalue.

(b) Let $SO(n)$ be the group of $n \times n$ rotation matrices. Let's check that the map $C : \mathfrak{so}(n) \to SO(n)$ is bijective onto the subset of $SO(n)$ that do not admit -1 as an eigenvalue.

To prove that C is injective, let $B_1, B_2 \in \mathfrak{so}(n)$, such that $C(B_1) = C(B_2)$, we have:

$$\begin{aligned}
C(B_1) &= C(B_2) \\
(I-B_1)(I+B_1)^{-1} &= (I-B_2)(I+B_2)^{-1} \\
(I-B_1)(I+B_1)^{-1}(I+B_1) &= (I-B_2)(I+B_2)^{-1}(I+B_1) \\
(I-B_1) &= (I-B_2)(I+B_2)^{-1}(I+B_1) \\
(I-B_1) &= (I+(-B_2))(I-(-B_2))^{-1}(I+B_1) \\
(I-B_1) &= (I-(-B_2))^{-1}(I+(-B_2))(I+B_1) \\
(I-B_1) &= (I+B_2)^{-1}(I-B_2)(I+B_1) \\
(I-B_1) &= (I+B_2)^{-1}(I-B_2+B_1-B_2B_1) \\
(I+B_2)(I-B_1) &= (I+B_2)(I+B_2)^{-1}(I-B_2+B_1-B_2B_1) \\
(I+B_2)(I-B_1) &= I-B_2+B_1-B_2B_1 \\
I+B_2-B_1-B_2B_1 &= I-B_2+B_1-B_2B_1 \\
2B_2 &= 2B_1 \\
B_2 &= B_1,
\end{aligned}$$

this proves that C is injective.

To prove that C is surjective onto the subset of rotation matrices that do not admit -1 as an eigenvalue, let $X \in SO(n)$, and X does not admit -1 as an eigenvalue, let's try to find an $B \in \mathfrak{so}(n)$ such that $C(B) = X$. First, let's see that $I + X$ is invertible, in fact, suppose it is not invertible, then there is a vector $v \in \mathbb{R}^n$, $v \neq 0$, such that $(I + X)v = 0$, so

$$\begin{aligned}
(I+X)v &= 0 \\
v + Xv &= 0 \\
Xv &= -v,
\end{aligned}$$

but this means that -1 is an eigenvalue of X, which is a contradiction with the definition of X.

Then we have:

$$
\begin{aligned}
(I-B)(I+B)^{-1} &= X \\
(I-B)(I+B)^{-1}(I+B) &= X(I+B) \\
I-B &= X(I+B) \\
I-B &= X+XB \\
I-B &= X+XB \\
I-X &= B+XB \\
I-X &= (I+X)B \\
(I+X)^{-1}(I-X) &= (I+X)^{-1}(I+X)B \\
(I+X)^{-1}(I-X) &= B.
\end{aligned}
$$

Observe also that if we use a property proved in 4(a) we have

$$
\begin{aligned}
(I-B)(I+B)^{-1} &= X \\
(I+(-B))(I-(-B))^{-1} &= X \\
(I-(-B))^{-1}(I+(-B)) &= X \\
(I+B)^{-1}(I-B) &= X \\
(I+B)(I+B)^{-1}(I-B) &= (I+B)X \\
I-B &= (I+B)X \\
I-B &= X+BX \\
I-X &= B+BX \\
I-X &= B(I+X) \\
(I-X)(I+X)^{-1} &= B(I+X)(I+X)^{-1} \\
(I-X)(I+X)^{-1} &= B,
\end{aligned}
$$

We can show that $B \in \mathfrak{so}(n)$, in fact:

$$
\begin{aligned}
B^T &= ((I+X)^{-1}(I-X))^T \\
&= (I-X)^T((I+X)^{-1})^T \\
&= (I-X)^T((I+X)^T)^{-1} \\
&= (I-X^T)((I+X^T))^{-1} \\
&= (I-X^{-1})((I+X^{-1}))^{-1} && \text{(because } X \text{ is a rotation matrix)} \\
&= (X-I)X^{-1}((X+I)X^{-1})^{-1} \\
&= (X-I)X^{-1}(X^{-1})^{-1}(X+I)^{-1} \\
&= (X-I)X^{-1}X(X+I)^{-1} \\
&= (X-I)(X+I)^{-1} \\
&= -(I-X)(X+I)^{-1} \\
&= -B,
\end{aligned}
$$

so $B \in \mathfrak{so}(n)$.

This means that C is surjective onto the space of rotation matrices that do not admit -1 as an eigenvalue, and the inverse of C is given by:

$$C^{-1}(X) = (I - X)(I + X)^{-1} = (I + X)^{-1}(I - X),$$

where $X \in SO(n)$, and X does not admits -1 as an eigenvalue.

Let's define the functions $f : \mathfrak{so}(n) \to SO(n), f(B) = I - B$ and $k : \mathfrak{so}(n) \to SO(n), k(B) = I + B$. The functions f and k are clearly continuous (we will calculate their derivatives soon). We saw in problem 3 that the function $h(A) = A^{-1}$ is differentiable in the space of invertible matrices, so it is continuous. Then the composition $g(B) = h \circ k(B)$ is continuous. The product of continuous functions is also continuous, then $C(B) = f(B)g(B)$ is continuous. Using a similar argument, we see that the inverse C^{-1} is also continuous. So we have shown that C is a continuous bijection from $\mathfrak{so}(n)$ to $C(\mathfrak{so}(n))$, with a continuous inverse, then we conclude that C is a homeomorphism between $\mathfrak{so}(n)$ and $C(\mathfrak{so}(n))$.

(c) If $f : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ and $g : M_n(\mathbb{R}) \to M_n(\mathbb{R})$ are differentiable matrix functions, lets calculate $D(fg)_A(B)$. We know that:

$$\begin{aligned}
f(A+B) &= f(A) + D(f)_A(B) + \epsilon_f(B)\|B\| \\
g(A+B) &= g(A) + D(g)_A(B) + \epsilon_g(B)\|B\|,
\end{aligned}$$

where $\lim_{B\to 0} \epsilon_f(B) = \lim_{B\to 0} \epsilon_f(B) = 0$.

So we have:

$$\begin{aligned}
fg(A+B) &= f(A+B)g(A+B) \\
&= \Big(f(A) + D(f)_A(B) + \epsilon_f(B)\|B\|\Big)\Big(g(A) + D(g)_A(B) + \epsilon_g(B)\|B\|\Big) \\
&= \Big(f(A) + D(f)_A(B) + \epsilon_f(B)\|B\|\Big)g(A) \\
&\quad + \Big(f(A) + D(f)_A(B) + \epsilon_f(B)\|B\|\Big)D(g)_A(B) \\
&\quad + \Big(f(A) + D(f)_A(B) + \epsilon_f(B)\|B\|\Big)\epsilon_g(B)\|B\| \\
&= f(A)g(A) + D(f)_A(B)g(A) + \epsilon_f(B)\|B\|g(A) \\
&\quad + f(A)D(g)_A(B) + D(f)_A(B)D(g)_A(B) + \epsilon_f(B)\|B\|D(g)_A(B) \\
&\quad + f(A)\epsilon_g(B)\|B\| + D(f)_A(B)\epsilon_g(B)\|B\| + \epsilon_f(B)\|B\|\epsilon_g(B)\|B\| \\
&= (fg)(A) + \Big(D(f)_A(B)g(A) + f(A)D(g)_A(B)\Big) \\
&\quad + \epsilon_f(B)\|B\|g(A) + D(f)_A(B)D(g)_A(B) + \epsilon_f(B)\|B\|D(g)_A(B) \\
&\quad + f(A)\epsilon_g(B)\|B\| + D(f)_A(B)\epsilon_g(B)\|B\| + \epsilon_f(B)\|B\|\epsilon_g(B)\|B\| \\
&= (fg)(A) + \Big(D(f)_A(B)g(A) + f(A)D(g)_A(B)\Big) \\
&\quad + \Big(\epsilon_f(B)g(A) + D(f)_A(B/\|B\|)D(g)_A(B) + \epsilon_f(B)D(g)_A(B) \\
&\quad + f(A)\epsilon_g(B) + D(f)_A(B)\epsilon_g(B) + \epsilon_f(B)\|B\|\epsilon_g(B)\Big)\|B\|.
\end{aligned}$$

Define then

$$\begin{aligned}
L(B) &= D(f)_A(B)g(A) + f(A)D(g)_A(B),\\
\epsilon_{fg}(B) &= \epsilon_f(B)g(A) + D(f)_A(B/\|B\|)D(g)_A(B) + \epsilon_f(B)D(g)_A(B)\\
&\quad + f(A)\epsilon_g(B) + D(f)_A(B)\epsilon_g(B) + \epsilon_f(B)\|B\|\epsilon_g(B)
\end{aligned}$$

From the linearity of $D(f)_A$ and $D(g)_A$, it follows that $L(B)$ is linear:

$$\begin{aligned}
L(B_1 + B_2) &= D(f)_A(B_1 + B_2)g(A) + f(A)D(g)_A(B_1 + B_2)\\
&= (D(f)_A(B_1) + D(f)_A(B_2))g(A) + f(A)(D(g)_A(B_1) + D(g)_A(B_2))\\
&= D(f)_A(B_1)g(A) + D(f)_A(B_2)g(A) + f(A)D(g)_A(B_1) + f(A)D(g)_A(B_2)\\
&= \Big(D(f)_A(B_1)g(A) + f(A)D(g)_A(B_1)\Big) + \Big(D(f)_A(B_2)g(A) + f(A)D(g)_A(B_2)\Big)\\
&= L(B_1) + L(B_2),
\end{aligned}$$

$$\begin{aligned}
L(\alpha B) &= D(f)_A(\alpha B)g(A) + f(A)D(g)_A(\alpha B)\\
&= \alpha D(f)_A(B)g(A) + \alpha f(A)D(g)_A(B)\\
&= \alpha(D(f)_A(B)g(A) + f(A)D(g)_A(B))\\
&= \alpha L(B).
\end{aligned}$$

Let's observe the behaviour of $\epsilon_{fg}(B)$ as B goes to zero:

$$\begin{aligned}
\|\epsilon_{fg}(B)\| &= \|\epsilon_f(B)g(A) + D(f)_A(B/\|B\|)D(g)_A(B) + \epsilon_f(B)D(g)_A(B)\\
&\quad + f(A)\epsilon_g(B) + D(f)_A(B)\epsilon_g(B) + \epsilon_f(B)\|B\|\epsilon_g(B)\|\\
&\leq \|\epsilon_f(B)g(A)\| + \|D(f)_A(B/\|B\|)D(g)_A(B)\| + \|\epsilon_f(B)D(g)_A(B)\|\\
&\quad + \|f(A)\epsilon_g(B)\| + \|D(f)_A(B)\epsilon_g(B)\| + \|\epsilon_f(B)\|B\|\epsilon_g(B)\|\\
&\leq \|\epsilon_f(B)\|\|g(A)\| + \|D(f)_A\|\|(B/\|B\|)\|\|D(g)_A\|\|B\| + \|\epsilon_f(B)\|\|D(g)_A\|\|B\|\\
&\quad + \|f(A)\|\|\epsilon_g(B)\| + \|D(f)_A\|\|B\|\|\epsilon_g(B)\| + \|\epsilon_f(B)\|\|B\|\|\epsilon_g(B)\|.
\end{aligned}$$

We know that $\lim_{B\to 0}\|B\| = \lim_{B\to 0}\|\epsilon_f(B)\| = \lim_{B\to 0}\|\epsilon_g(B)\| = 0$, then

$$\begin{aligned}
\lim_{B\to 0}\|\epsilon_{fg}(B)\| &\leq \lim_{B\to 0}\Big(\|\epsilon_f(B)\|\|g(A)\| + \|D(f)_A\|\|(B/\|B\|)\|\|D(g)_A\|\|B\|\\
&\qquad + \|\epsilon_f(B)\|\|D(g)_A\|\|B\| + \|f(A)\|\|\epsilon_g(B)\|\\
&\qquad + \|D(f)_A\|\|B\|\|\epsilon_g(B)\| + \|\epsilon_f(B)\|\|B\|\|\epsilon_g(B)\|\Big)\\
&= 0.
\end{aligned}$$

So $\lim_{B\to 0}\epsilon_{fg}(B) = 0$.

Then we can write

$$fg(A + B) = fg(A) + L(B) + \epsilon_{fg}(B)\|B\|,$$

where $L(B)$ is linear, and $\lim_{B\to 0}\epsilon_{fg}(B) = 0$, so we have

$$D(fg)_A(B) = L(B) = D(f)_A(B)g(A) + f(A)D(g)_A(B).$$

(d) Let's calculate $DC(B)(A)$. We know that $C(B) = (I-B)(I+B)^{-1}$, we can define $f(B) = I - B$, and $g(B) = (I+B)^{-1}$, so $C(B) = f(B)g(B)$, and we can use the result obtained above:

$$DC(B)(A) = D(fg)_B(A) = D(f)_B(A)g(B) + f(B)D(g)_B(A).$$

It is easy to see that $D(f)_B(A) = -A$, in fact, we can write

$$f(B+A) = I - (B+A) = I - B - A = f(B) - A = f(B) + L(A) + \epsilon_f(A)\|A\|,$$

where $L(A) = -A$, $\epsilon_f(A) = 0$, so $\lim_{A\to 0} \epsilon_f(A) = 0$, then $D(f)_B(A) = L(A) = -A$.

Similarly, for a function defined as $k(B) = I + B$ (which we will use below), we have $D(k)_B(A) = A$.

To calculate $D(g)_B(A)$ we can use known results:

$$\begin{aligned}
h(B) &:= B^{-1} \\
k(B) &:= I + B \\
g(B) &= (h \circ k)(B) \\
D(g)_B(A) &= \big(D(h)_{k(B)} \circ D(k)_B\big)(A) && \text{(chain rule)} \\
D(g)_B(A) &= D(h)_{k(B)}(D(k)_B(A)) \\
D(g)_B(A) &= D(h)_{k(B)}(A) \\
&= -(k(B))^{-1}A(k(B)) && \text{(using result from problem 3)} \\
&= -(I+B)^{-1}A(I+B)^{-1}.
\end{aligned}$$

Then we can calculate

$$\begin{aligned}
DC(B)(A) &= D(f)_B(A)g(B) + f(B)D(g)_B(A) \\
&= (-A)(I+B)^{-1} + (I-B)(-(I+B)^{-1}A(I+B)^{-1}) \\
&= -A(I+B)^{-1} - (I-B)(I+B)^{-1}A(I+B)^{-1} \\
&= -(A(I+B)^{-1} + (I-B)(I+B)^{-1}A(I+B)^{-1}) \\
&= -(I + (I-B)(I+B)^{-1})A(I+B)^{-1}
\end{aligned}$$

Let $B \in \mathfrak{so}(n)$, $A_1, A_2 \in M_n(\mathbb{R})$, such that $dC(B)(A_1) = dC(B)(A_2)$, we have then

$$\begin{aligned}
dC(B)(A_1) &= dC(B)(A_2) \\
-(I + (I-B)(I+B)^{-1})A_1(I+B)^{-1} &= -(I + (I-B)(I+B)^{-1})A_2(I+B)^{-1} \\
(I + (I-B)(I+B)^{-1})A_1(I+B)^{-1}(I+B) &= (I + (I-B)(I+B)^{-1})A_2(I+B)^{-1}(I+B) \\
(I + (I-B)(I+B)^{-1})A_1 &= (I + (I-B)(I+B)^{-1})A_2 \\
((I+B) + (I-B))(I+B)^{-1}A_1 &= ((I+B) + (I-B))(I+B)^{-1}A_2 \\
2(I+B)^{-1}A_1 &= 2(I+B)^{-1}A_2 \\
(I+B)(I+B)^{-1}A_1 &= (I+B)(I+B)^{-1}A_2 \\
A_1 &= A_2,
\end{aligned}$$

so $dC(B)$ is injective.

We saw that C is a homeomorphism between $\mathfrak{so}(n)$ and $C(\mathfrak{so}(n))$, and $dC(B)$ is injective for every $B \in \mathfrak{so}(n)$. The space $\mathfrak{so}(n)$ is isomorphic to $\mathbb{R}^{n(n-1)/2}$, so it is an open subset of $M_n(\mathbb{R})$. Then C satisfies all conditions to be a parametrization.

5 Problem 5

Consider the parametric surface given by

$$\begin{aligned} x(u,v) &= \frac{8uv}{(u^2+v^2+1)^2}, \\ y(u,v) &= \frac{4v(u^2+v^2-1)}{(u^2+v^2+1)^2}, \\ z(u,v) &= \frac{4(u^2-v^2)}{(u^2+v^2+1)^2}. \end{aligned}$$

Let's make the change of variables

$$\begin{aligned} u &= \rho\cos\theta \\ v &= \rho\sin\theta. \end{aligned}$$

We have then

$$\begin{aligned} x(\rho,\theta) &= \frac{8(\rho\cos\theta)(\rho\sin\theta)}{((\rho\cos\theta)^2+(\rho\sin\theta)^2+1)^2} \\ &= \frac{8\rho^2\cos\theta\sin\theta}{(\rho^2\cos^2\theta+\rho^2\sin^2\theta+1)^2} \\ &= \frac{8\rho^2\cos\theta\sin\theta}{(\rho^2+1)^2} \\ &= \frac{4\rho^2\cdot 2\cos\theta\sin\theta}{(\rho^2+1)^2} \\ &= \frac{4\rho^2\sin 2\theta}{(\rho^2+1)^2}, \end{aligned}$$

and

$$\begin{aligned} y(\rho,\theta) &= \frac{4(\rho\sin\theta)((\rho\cos\theta)^2+(\rho\sin\theta)^2-1)}{((\rho\cos\theta)^2+(\rho\sin\theta)^2+1)^2} \\ &= \frac{4(\rho\sin\theta)(\rho^2-1)}{(\rho^2+1)^2} \\ &= \frac{4\rho(\rho^2-1)}{(\rho^2+1)^2}\sin\theta, \end{aligned}$$

and

$$\begin{aligned} z(\rho,\theta) &= \frac{4((\rho\cos\theta)^2-(\rho\sin\theta)^2)}{((\rho\cos\theta)^2+(\rho\sin\theta)^2+1)^2} \\ &= \frac{4\rho^2(\cos^2\theta-\sin^2\theta)}{(\rho^2+1)^2} \\ &= \frac{4\rho^2\cos 2\theta}{(\rho^2+1)^2}. \end{aligned}$$

Observe that for $\rho \in [0,1]$ and for any θ, we can find an angle $\theta_0 = \theta + 2k\pi$, for an integer k, such that $\theta_0 \in [-\pi,\pi]$, so we will have $\sin(\theta_0) = \sin(\theta + 2k\pi) = \sin(\theta)$,

$\sin(2\theta_0) = \sin(2\theta + 4k\pi) = \sin(2\theta)$ and $\cos(2\theta_0) = \cos(2\theta + 4k\pi) = \cos(2\theta)$, then $x(\rho, \theta) = x(\rho, \theta_0)$, $y(\rho, \theta) = y(\rho, \theta_0)$ and $z(\rho, \theta) = z(\rho, \theta_0)$.

When $\rho \neq 0$, we can define $\rho_0 = 1/\rho$, so we have:

$$\begin{aligned} x(\rho_0, \theta) &= \frac{4(1/\rho)^2 \sin 2\theta}{((1/\rho)^2 + 1)^2} \\ &= \frac{1}{\rho^2} \frac{4}{((1/\rho)^2(1 + \rho^2))^2} \sin 2\theta \\ &= \frac{1}{\rho^2} \frac{4}{(1/\rho)^4(1 + \rho^2)^2} \sin 2\theta \\ &= \frac{4}{(1/\rho)^2(1 + \rho^2)^2} \sin 2\theta \\ &= \frac{4\rho^2}{(1 + \rho^2)^2} \sin 2\theta \\ &= x(\rho, \theta). \end{aligned}$$

For y we have:

$$\begin{aligned} y(\rho_0, \theta) &= \frac{4(1/\rho)((1/\rho)^2 - 1)}{((1/\rho)^2 + 1)^2} \sin\theta \\ &= \frac{4(1/\rho)(1/\rho)^2(1 - \rho^2)}{((1/\rho)^2(1 + \rho^2))^2} \sin\theta \\ &= \frac{4(1/\rho)^3(1 - \rho^2)}{(1/\rho)^4(1 + \rho^2)^2} \sin\theta \\ &= \frac{4(1 - \rho^2)}{(1/\rho)(1 + \rho^2)^2} \sin\theta \\ &= \frac{4\rho(1 - \rho^2)}{(1 + \rho^2)^2} \sin\theta \\ &= -\frac{4\rho(\rho^2 - 1)}{(1 + \rho^2)^2} \sin\theta \\ &= -y(\rho, \theta). \end{aligned}$$

For z we have:

$$\begin{aligned} z(\rho_0, \theta) &= \frac{4(1/\rho)^2}{((1/\rho)^2 + 1)^2} \cos 2\theta \\ &= \frac{1}{\rho^2} \frac{4}{((1/\rho)^2(1 + \rho^2))^2} \cos 2\theta \\ &= \frac{1}{\rho^2} \frac{4}{(1/\rho)^4(1 + \rho^2)^2} \cos 2\theta \\ &= \frac{4}{(1/\rho)^2(1 + \rho^2)^2} \cos 2\theta \\ &= \frac{4\rho^2}{(1 + \rho^2)^2} \cos 2\theta \\ &= z(\rho, \theta). \end{aligned}$$

Using $\theta_0 = \theta + 2k\pi - \pi$, for an integer k such that $\theta_0 \in [-\pi, \pi]$, we have

$$\begin{aligned}
x(\rho_0, \theta_0) &= \frac{4\rho_0^2}{(1+\rho_0^2)^2} \sin 2\theta_0 \\
&= \frac{4\rho_0^2}{(1+\rho_0^2)^2} \sin 2(\theta + 2k\pi - \pi) \\
&= \frac{4\rho_0^2}{(1+\rho_0^2)^2} \sin(2(\theta - \pi) + 4k\pi) \\
&= \frac{4\rho_0^2}{(1+\rho_0^2)^2} \sin(2(\theta - \pi)) \\
&= \frac{4\rho_0^2}{(1+\rho_0^2)^2} \sin(2\theta - 2\pi) \\
&= \frac{4\rho_0^2}{(1+\rho_0^2)^2} \sin(2\theta) \\
&= x(\rho_0, \theta) \\
&= x(\rho, \theta),
\end{aligned}$$

$$\begin{aligned}
y(\rho_0, \theta_0) &= \frac{4\rho_0(\rho_0^2 - 1)}{(1+\rho_0^2)^2} \sin \theta_0 \\
&= \frac{4\rho_0(\rho_0^2 - 1)}{(1+\rho_0^2)^2} \sin(\theta - \pi + 2k\pi) \\
&= \frac{4\rho_0(\rho_0^2 - 1)}{(1+\rho_0^2)^2} \sin(\theta - \pi) \\
&= \frac{4\rho_0(\rho_0^2 - 1)}{(1+\rho_0^2)^2} (-\sin \theta) \\
&= -\frac{4\rho_0(\rho_0^2 - 1)}{(1+\rho_0^2)^2} \sin \theta \\
&= -y(\rho_0, \theta) \\
&= y(\rho, \theta),
\end{aligned}$$

$$\begin{aligned}
z(\rho_0, \theta_0) &= \frac{4\rho_0^2}{(1+\rho_0^2)^2} \cos 2\theta_0 \\
&= \frac{4\rho_0^2}{(1+\rho_0^2)^2} \cos(2(\theta - \pi - 2k\pi)) \\
&= \frac{4\rho_0^2}{(1+\rho_0^2)^2} \cos(2\theta - 2\pi - 4k\pi) \\
&= \frac{4\rho_0^2}{(1+\rho_0^2)^2} \cos 2\theta \\
&= z(\rho_0, \theta) \\
&= z(\rho, \theta).
\end{aligned}$$

Then for $\rho > 1$ and any θ, we can find $\rho_0 \in [0, 1]$ and $\theta_0 \in [-\pi, \pi]$ by $\rho_0 = 1/\rho$, and $\theta_0 = \theta - \pi + 2k\pi$, such that $x(\rho, \theta) = x(\rho_0, \theta_0)$, $y(\rho, \theta) = y(\rho_0, \theta_0)$ and $z(\rho, \theta) = z(\rho_0, \theta_0)$.

As we are working with polar coordinates, we only use $\rho \geq 0$, so it is not necessary to investigate what happens when $\rho < 0$. This conclude s that the entire trace of this surface can be obtained using $\rho \in [0, 1]$ and $\theta \in [-\pi, \pi]$.

This surface, using this parameterization is ploted in Figure 1.

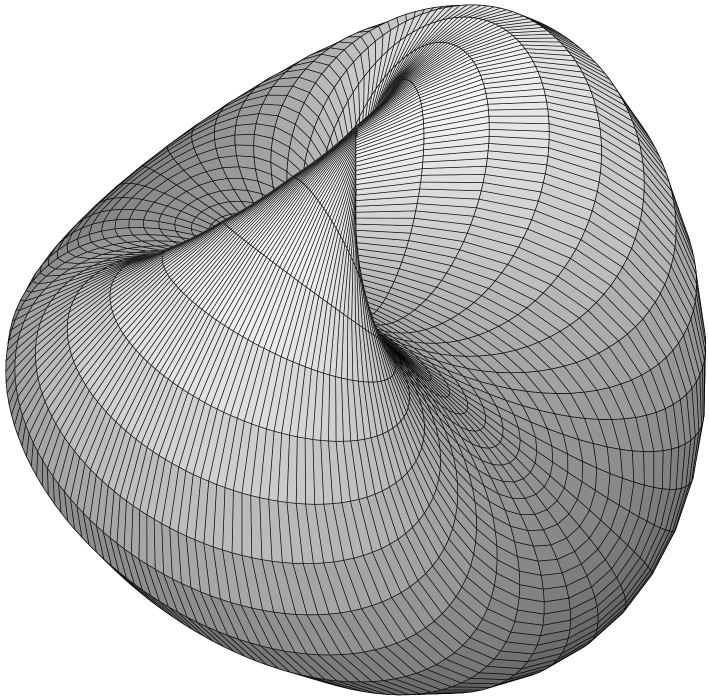

Figure 1: Plot of the surface.

Observe that when we fix $\theta = 0$ we have

$$x(\rho, 0) = 0$$
$$y(\rho, 0) = 0$$
$$z(\rho, 0) = \frac{4\rho^2}{(\rho^2 + 1)^2}.$$

For $\theta = \pi$ we have

$$x(\rho, \pi) = 0$$
$$y(\rho, \pi) = 0$$
$$z(\rho, \pi) = \frac{4\rho^2}{(\rho^2 + 1)^2}.$$

The same happens to $\theta = -\pi$:

$$x(\rho, -\pi) = 0$$
$$y(\rho, -\pi) = 0$$
$$z(\rho, -\pi) = \frac{4\rho^2}{(\rho^2 + 1)^2}.$$

So the surface self-intersects for $\theta \in \{-\pi, 0, \pi\}$, in the segment $(0, 0, \alpha(\rho))$, where

$$\alpha(\rho) = \frac{4\rho^2}{(\rho^2+1)^2}.$$

It is easy to see that $\alpha(0) = 0$, $\alpha(1) = 1$, and it is non-decrescent for $\rho \in [0, 1]$, which can be seen by its graph, in Figure 2.

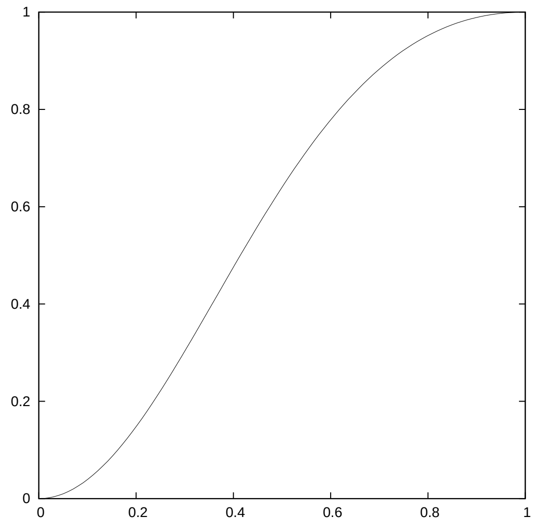

Figure 2: Graph of function $\alpha(\rho)$.

The point corresponding to $\rho = 1$ and $\theta = 0$ on the surface is the point $(0, 0, 1)$, which is the same point obtained by using $\rho = 1$ and $\theta = \pi$, and also by using $\rho = 1$ and $\theta = -\pi$, according to the analysis above.

In Figure 3 we plot this surface using $\theta \in [0, \pi]$ and $\rho \in [0, 1]$.

(b) Let's express the trigonometric functions in terms of $u = \tan(\theta/2)$, and let $v = \rho$. Observe that

$$\tan\theta = \tan\left(2\frac{\theta}{2}\right) = \frac{2\tan(\theta/2)}{1-\tan^2(\theta/2)} = \frac{2u}{1-u^2},$$

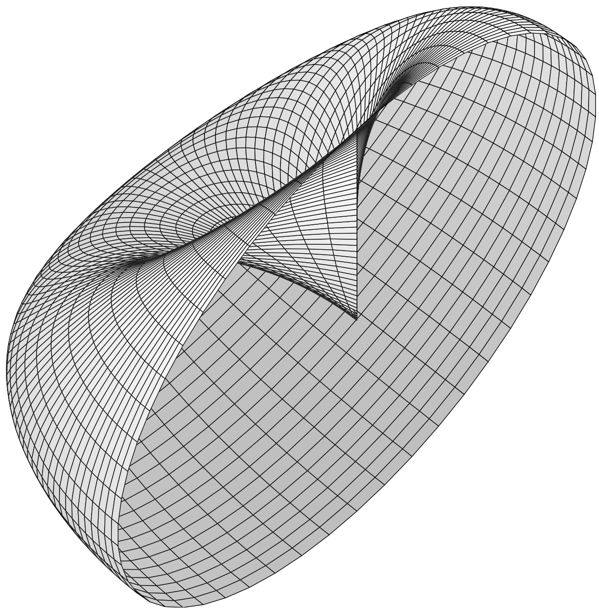

Figure 3: Plot of the surface using $\theta \in [0, \pi]$ and $\rho \in [0, 1]$.

$$
\begin{aligned}
\sin 2\theta &= \frac{2\tan\theta}{1+\tan^2\theta} \\
&= 2\left(\frac{2u}{1-u^2}\right)\left(\frac{1}{1+\left(\frac{2u}{1-u^2}\right)^2}\right) \\
&= \frac{4u}{(1-u^2)\left(1+\left(\frac{2u}{1-u^2}\right)^2\right)} \\
&= \frac{4u}{\left((1-u^2)+(1-u^2)\left(\frac{2u}{1-u^2}\right)^2\right)} \\
&= \frac{4u}{\left((1-u^2)+\frac{4u^2}{1-u^2}\right)} \\
&= \frac{4u}{\left(\frac{(1-u^2)^2+4u^2}{1-u^2}\right)} \\
&= \frac{4u}{\left(\frac{1-2u^2+u^4+4u^2}{1-u^2}\right)} \\
&= \frac{4u}{\left(\frac{1+2u^2+u^4}{1-u^2}\right)} \\
&= \frac{4u}{\left(\frac{(1+u^2)^2}{1-u^2}\right)} \\
&= \frac{4u(1-u^2)}{(1+u^2)^2},
\end{aligned}
$$

then

$$
\begin{aligned}
x(u,v) &= \frac{4\rho^2}{(\rho^2+1)^2}\sin 2\theta \\
&= \frac{4v^2}{(v^2+1)^2}\frac{4u(1-u^2)}{(1+u^2)^2} \\
&= \frac{16u(1-u^2)v^2}{(v^2+1)^2(1+u^2)^2}.
\end{aligned}
$$

Observe now that

$$
\sin\theta = \sin\left(2\frac{\theta}{2}\right) = \frac{2\tan(\theta/2)}{1+\tan^2(\theta/2)} = \frac{2u}{1+u^2} = \frac{2u(1+u^2)}{(1+u^2)^2}
$$

so we have

$$
\begin{aligned}
y(u,v) &= \frac{4\rho(\rho^2-1)}{(\rho^2+1)^2}\sin\theta \\
&= \frac{4v(v^2-1)}{(v^2+1)^2}\frac{2u(1+u^2)}{(1+u^2)^2} \\
&= \frac{8uv(v^2-1)(1+u^2)}{(v^2+1)^2(1+u^2)^2}
\end{aligned}
$$

And finally

$$
\begin{aligned}
\cos 2\theta &= 1-2\sin^2\theta \\
&= 1-2\left(\frac{2u}{1+u^2}\right)^2 \\
&= 1-\frac{8u^2}{(1+u^2)^2} \\
&= \frac{(1+u^2)^2-8u^2}{(1+u^2)^2} \\
&= \frac{1+2u^2+u^4-8u^2}{(1+u^2)^2} \\
&= \frac{1-6u^2+u^4}{(1+u^2)^2},
\end{aligned}
$$

then we have

$$
\begin{aligned}
z(u,v) &= \frac{4\rho^2}{(\rho^2+1)^2}\cos 2\theta \\
&= \frac{4v^2}{(v^2+1)^2}\frac{1-6u^2+u^4}{(1+u^2)^2} \\
&= \frac{4v^2(1-6u^2+u^4)}{(v^2+1)^2(1+u^2)^2}.
\end{aligned}
$$