Computational Manifolds and Applications–2011, IMPA

Homework 2

Due September 29, 2011

Problem 1. Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be the function given by

$$f(x,y) = \begin{cases} \frac{x^2 y}{x^4 + y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

(a) Compute the directional derivative $\mathrm{D}_u f(0,0)$ of f at $(0,0)$ for every vector $u = (u_1, u_2) \neq 0$.

(b) Prove that the derivative $\mathrm{D}f(0,0)$ does not exist. What is the behavior of the function f on the parabola $y = x^2$ near the origin $(0,0)$?

Problem 2. (a) Let $f\colon \mathrm{M}_n(\mathbb{R}) \to \mathrm{M}_n(\mathbb{R})$ be the function defined on $n \times n$ matrices by

$$f(A) = A^2.$$

Prove that

$$\mathrm{D}f_A(H) = AH + HA,$$

for all $A, H \in \mathrm{M}_n(\mathbb{R})$.

(b) Let $f\colon \mathrm{M}_n(\mathbb{R}) \to \mathrm{M}_n(\mathbb{R})$ be the function defined on $n \times n$ matrices by

$$f(A) = A^3.$$

Prove that

$$\mathrm{D}f_A(H) = A^2 H + AHA + HA^2,$$

for all $A, H \in \mathrm{M}_n(\mathbb{R})$.

Problem 3. Let $f\colon \mathrm{GL}(n,\mathbb{R}) \to \mathrm{M}_n(\mathbb{R})$ be the function defined on invertible $n \times n$ matrices by

$$f(A) = A^{-1}.$$

Prove that

$$\mathrm{D}f_A(H) = -A^{-1} H A^{-1},$$

for all $A \in \mathrm{GL}(n,\mathbb{R})$ and for all $H \in \mathrm{M}_n(\mathbb{R})$.

Problem 4. Recall that a matrix $B \in \mathrm{M}_n(\mathbb{R})$ is skew-symmetric if

$$B^\top = -B.$$

Check that the set $\mathfrak{so}(n)$ of skew-symmetric matrices is a vector space of dimension $n(n-1)/2$, and thus is isomorphic to $\mathbb{R}^{n(n-1)/2}$. Let $C\colon \mathfrak{so}(n) \to \mathrm{M}_n(\mathbb{R})$ be the function given by

$$C(B) = (I - B)(I + B)^{-1}.$$

Prove that the eigenvalues of a skew-symmetric matrix are either 0 or pure imaginary (that is, of the form $i\mu$ for $\mu \in \mathbb{R}$.). Prove that if B is skew-symmetric, then $I - B$ and $I + B$ are invertible, and so C is well-defined. Prove that

$$(I + B)(I - B) = (I - B)(I + B),$$

and that

$$(I + B)(I - B)^{-1} = (I - B)^{-1}(I + B).$$

Prove that

$$(C(B))^\top C(B) = I$$

and that

$$\det C(B) = +1,$$

so that $C(B)$ is a rotation matrix. Furthermore, show that $C(B)$ does not admit -1 as an eigenvalue.

(b) Let $\mathbf{SO}(n)$ be the group of $n \times n$ rotation matrices. Prove that the map

$$C\colon \mathfrak{so}(n) \to \mathbf{SO}(n)$$

is bijective onto the subset of rotation matrices that do not admit -1 as an eigenvalue. Show that the inverse of this map is given by

$$B = (I + R)^{-1}(I - R) = (I - R)(I + R)^{-1},$$

where $R \in \mathbf{SO}(n)$ does not admit -1 as an eigenvalue. Check that C is a homeomorphism between $\mathfrak{so}(n)$ and $C(\mathfrak{so}(n))$.

(c) If $f\colon \mathrm{M}_n(\mathbb{R}) \to \mathrm{M}_n(\mathbb{R})$ and $g\colon \mathrm{M}_n(\mathbb{R}) \to \mathrm{M}_n(\mathbb{R})$ are differentiable matrix functions, prove that

$$\mathrm{D}(fg)_A(B) = f(A)\mathrm{D}(g)_A(B) + \mathrm{D}(f)_A(B)g(A),$$

for all $A, B \in \mathrm{M}_n(\mathbb{R})$.

(d) Prove that

$$dC(B)(A) = -[I + (I - B)(I + B)^{-1}]A(I + B)^{-1}.$$

Prove that $dC(B)$ is injective, for every skew-symmetric matrix B. Prove that C a parametrization of $\mathbf{SO}(n)$.

Problem 5. Consider the parametric surface given by

$$\begin{aligned} x(u,v) &= \frac{8uv}{(u^2+v^2+1)^2}, \\ y(u,v) &= \frac{4v(u^2+v^2-1)}{(u^2+v^2+1)^2}, \\ z(u,v) &= \frac{4(u^2-v^2)}{(u^2+v^2+1)^2}. \end{aligned}$$

The trace of this surface is called a *crosscap*. In order to plot this surface, make the change of variables

$$\begin{aligned} u &= \rho\cos\theta \\ v &= \rho\sin\theta. \end{aligned}$$

Prove that we obtain the parametric definition

$$\begin{aligned} x &= \frac{4\rho^2}{(\rho^2+1)^2}\,\sin 2\theta, \\ y &= \frac{4\rho(\rho^2-1)}{(\rho^2+1)^2}\,\sin\theta, \\ z &= \frac{4\rho^2}{(\rho^2+1)^2}\,\cos 2\theta. \end{aligned}$$

Show that the entire trace of the surface is obtained for $\rho \in [0,1]$ and $\theta \in [-\pi,\pi]$.

Hint. What happens if you change ρ to $1/\rho$?

Plot the trace of the surface using the above parametrization. Show that there is a line of self-intersection along the portion of the z-axis corresponding to $0 \leq z \leq 1$. What can you say about the point corresponding to $\rho = 1$ and $\theta = 0$?

Plot the portion of the surface for $\rho \in [0,1]$ and $\theta \in [0,\pi]$.

(b) Express the trigonometric functions in terms of $u = \tan(\theta/2)$, and letting $v = \rho$, show that we get

$$\begin{aligned} x &= \frac{16uv^2(1-u^2)}{(u^2+1)^2(v^2+1)^2}, \\ y &= \frac{8uv(u^2+1)(v^2-1)}{(u^2+1)^2(v^2+1)^2}, \\ z &= \frac{4v^2(u^4-6u^2+1)}{(u^2+1)^2(v^2+1)^2}. \end{aligned}$$