Computational Manifolds and Applications - 2011

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1 Problem 1

In Figure 1 we ploted the curve defined by

$$f(t) = \begin{cases} (t, t^2 \sin(1/t)) & \text{if } t \neq 0; \\ (0,0) & \text{if } t = 0. \end{cases}$$

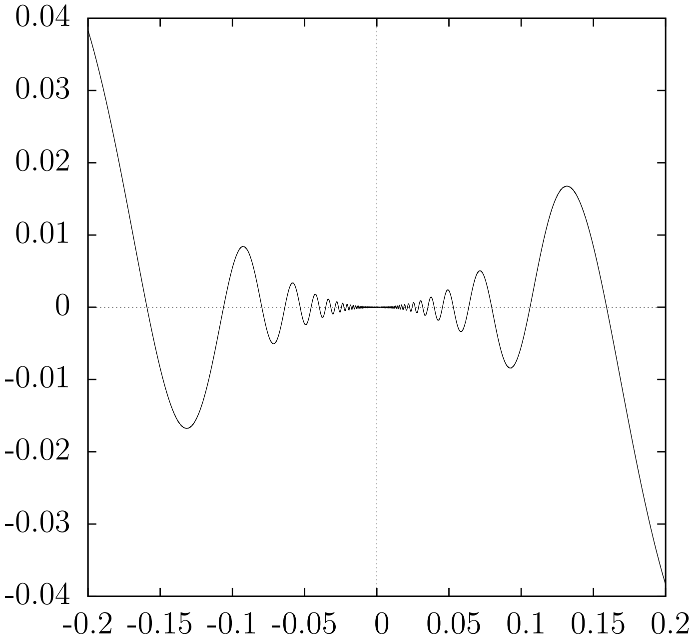

Figure 1: Function f.

We can calculate $f'(0)$ using the definition of derivative:

$$\begin{aligned} f'(0) &= \lim_{t\to 0} \frac{f(0+t) - f(0)}{t} \\ &= \lim_{t\to 0} \frac{f(t) - f(0)}{t} \\ &= \lim_{t\to 0} \frac{(t, t^2\sin(1/t)) - (0,0)}{t} \\ &= \lim_{t\to 0} (1, t\sin(1/t)) \\ &= (1,0) && \text{(because sin is bounded).} \end{aligned}$$

For $t \neq 0$ we have:

$$
\begin{aligned}
f'(t) &= \frac{d}{dt}(t, t^2 \sin(1/t)) \\
&= \left(1, 2t\sin(1/t) + t^2\cos(1/t)\frac{-1}{t^2}\right) \\
&= (1, 2t\sin(1/t) - \cos(1/t))\,.
\end{aligned}
$$

To see that f' is discontinuous at 0, we can use the sequence defined by $t_n = 1/(2n\pi)$, so we have:

$$
\lim_{n\to\infty} t_n = \lim_{n\to\infty} \frac{1}{2n\pi} = 0,
$$

and

$$
\begin{aligned}
f'(t_n) &= (1, 2t_n \sin(1/t_n) - \cos(1/t_n)) \\
&= \left(1, 2\frac{1}{2n\pi}\sin(2n\pi) - \cos(2n\pi)\right) \\
&= \left(1, 2\frac{1}{2n\pi}\cdot 0 - 1\right) \\
&= (1, -1)\,,
\end{aligned}
$$

then

$$
\lim_{n\to\infty} f'(t_n) = (1,-1) \neq f'\left(\lim_{n\to\infty} t_n\right) = f'(0) = (1,0),
$$

so f' is discontinuous.

2 Problem 2

A cardioid is the curve defined in polar coordinates (ρ, θ) by

$$
\rho = 1 + \cos\theta.
$$

(a) Transforming from polar coordinates to cartesian coordinates we have:

$$
\begin{aligned}
x &= \rho\cos\theta, \\
y &= \rho\sin\theta,
\end{aligned}
$$

so we can find a parametric description of the cardioid using $\rho = 1 + \cos\theta$:

$$
\begin{aligned}
x(\theta) &= (1+\cos\theta)\cos\theta, \\
y(\theta) &= (1+\cos\theta)\sin\theta.
\end{aligned}
$$

This curve is ploted in Figure 2, using $\theta \in [0, 2\pi]$.

To find the regular points, we first need to calculate the derivative of $f(\theta) = (x(\theta), y(\theta))$:

$$
\begin{aligned}
x'(\theta) &= -\sin\theta\cos\theta + (1+\cos\theta)(-\sin\theta) \\
&= -\sin\theta\cos\theta - \sin\theta - \sin\theta\cos\theta \\
&= -\sin\theta(2\cos\theta + 1)
\end{aligned}
$$

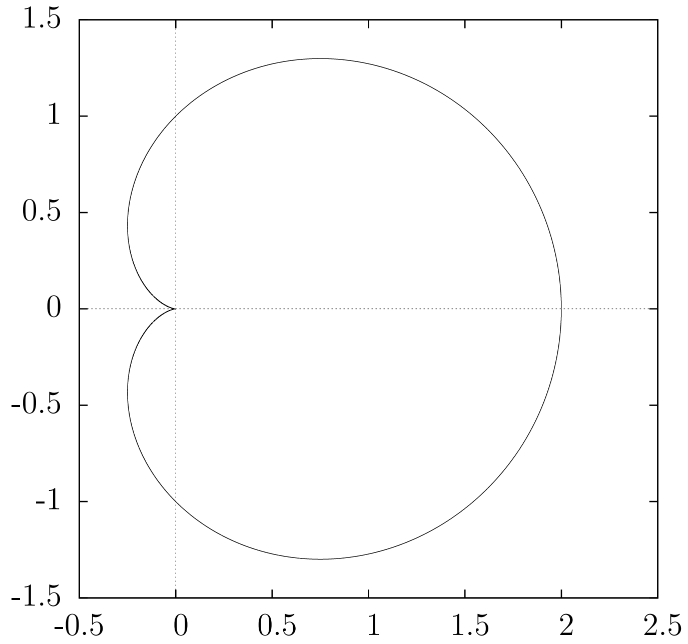

Figure 2: Cardioid.

$$
\begin{aligned}
y'(\theta) &= -\sin\theta\sin\theta + (1+\cos\theta)\cos\theta \\
&= -\sin^2\theta + \cos\theta + \cos^2\theta \\
&= \cos(2\theta) + \cos\theta \\
&= 2\cos^2\theta - 1 + \cos\theta
\end{aligned}
$$

When $x'(\theta) = 0$ we have $-\sin\theta(2\cos\theta + 1) = 0$, so $\sin\theta = 0$ or $2\cos\theta + 1 = 0$, in the first case $\theta = 0$ or $\theta = \pi$ (considering θ in $[0, 2\pi]$), in the second case $\cos\theta = -1/2$. For $\theta = 0$, $y'(\theta) = y'(0) = 2\cos^2(0) - 1 + \cos(0) = 2 - 1 + 1 = 2 \neq 0$.

When $\theta = \pi$, $y'(\theta) = y'(\pi) = 2\cos^2(\pi) - 1 + \cos(\pi) = 2(-1)^2 - 1 - 1 = 0$, then $f'(\pi) = (0, 0)$, so $(x(\pi), y(\pi)) = (0, 0)$ is a regular point.

For $\cos\theta = -1/2$, $y'(\theta) = 2\cos^2\theta - 1 + \cos\theta = 2(-1/2)^2 - 1 + (-1/2) = 2/4 - 1 - 1/2 = -1 \neq 0$, then $f'(\theta) \neq (0, 0)$, so this is not a regular point.

So only the origin (when $\theta = \pi$) is not a regular point of the cardioid.

(b) Let's prove that the cardioid is also defined by the equations

$$
\begin{aligned}
x &= \frac{2(1-t^2)}{(1+t^2)^2} \\
y &= \frac{4t}{(1+t^2)^2}.
\end{aligned}
$$

Changing (x, y) to polar coordinates (ρ, θ), we have

$$\begin{aligned}
\rho &= \sqrt{x^2 + y^2} \\
&= \sqrt{\left(\frac{2(1-t^2)}{(1+t^2)^2}\right)^2 + \left(\frac{4t}{(1+t^2)^2}\right)^2} \\
&= \sqrt{\frac{4(1-t^2)^2}{(1+t^2)^4} + \frac{16t^2}{(1+t^2)^4}} \\
&= \sqrt{\frac{4(1-t^2)^2 + 16t^2}{(1+t^2)^4}} \\
&= \sqrt{\frac{4(1-2t^2+t^4) + 16t^2}{(1+t^2)^4}} \\
&= \sqrt{\frac{4(1+2t^2+t^4)}{(1+t^2)^4}} \\
&= \sqrt{\frac{4(1+t^2)^2}{(1+t^2)^4}} \\
&= \frac{2}{1+t^2}
\end{aligned}$$

$$\begin{aligned}
x = \rho\cos\theta &= \frac{2(1-t)^2}{(1+t^2)^2} \\
\frac{2}{(1+t^2)^2}\cos\theta &= \frac{2(1-t)^2}{(1+t^2)^2} \\
\cos\theta &= \frac{2(1-t)^2}{(1+t^2)^2},
\end{aligned}$$

so

$$\begin{aligned}
1 + \cos\theta &= 1 + \frac{1-t^2}{1+t^2} \\
&= \frac{1+t^2+1-t^2}{1+t^2} \\
&= \frac{2}{1+t^2} \\
&= \rho.
\end{aligned}$$

This means that the points (x, y) belong to the cardioid.

We can now show a relation between the parameters θ and t. When $\theta \neq \pi$, then $\rho \neq 0$, so we can do

$$\begin{aligned}
\rho &= \frac{2}{1+t^2} \\
1 + t^2 &= 2/\rho \\
t &= \pm\sqrt{2/\rho - 1}.
\end{aligned}$$

The term $(2/\rho - 1)$ is always non-negative because $\rho \in [0, 2]$ (from $\rho = 1 + \cos\theta$). To choose the sign of t, we observe that $sign(t) = sign(y)$. For $\theta \in [0, \pi)$, we have $y = (1 + \cos\theta)\sin\theta \geq 0$, then $t = \sqrt{2/\rho - 1}$. For $\theta \in (\pi, 2\pi]$, we have $y = (1 + \cos\theta)\sin\theta \leq 0$, then $t = -\sqrt{2/\rho - 1}$. When $\theta = \pi$, we are in the point $(0, 0)$, which is

$$(0,0) = \lim_{t\to\infty} \left(\frac{2(1-t^2)}{(1+t^2)^2}, \frac{4t}{(1+t^2)^2} \right) = \lim_{t\to-\infty} \left(\frac{2(1-t^2)}{(1+t^2)^2}, \frac{4t}{(1+t^2)^2} \right),$$

this means that this point does not belong to this parameterization, but is a limit point as t increases (or decreases).

3 Problem 3

The *Descartes Folium* (a portion of this curve) is defined by the parametric curve $\alpha : (-1, \infty) \to \mathbb{R}^2$

$$\alpha(t) = \left(\frac{3t}{1+t^3}, \frac{3t^2}{1+t^3} \right).$$

(a) This curve is plotted in Figure 3.

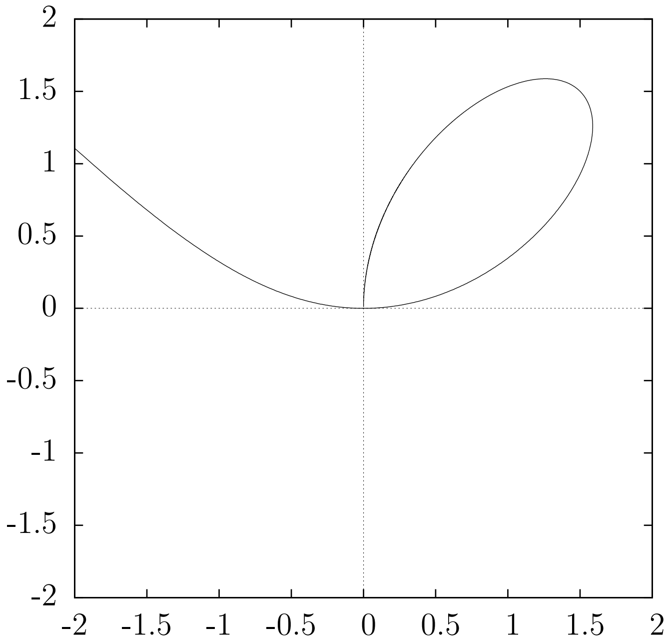

Figure 3: Descartes Folium.

(b) Let's prove that α is injective. Let $t_0, t_1 \in (-1, \infty)$ such that $\alpha(t_0) = \alpha(t_1)$. When $t_0 = 0$, $\alpha(t_0) = (0, 0) = \alpha(t_1)$, so $\frac{3t_1}{1+t_1^3} = 0$, then $t_1 = 0 = t_0$.

Now let $t_0 \neq 0$ and $t_1 \neq 0$:

$$\begin{aligned}
\alpha(t_0) &= \alpha(t_1) \\
\left(\frac{3t_0}{1+t_0^3}, \frac{3t_0^2}{1+t_0^3}\right) &= \left(\frac{3t_1}{1+t_1^3}, \frac{3t_1^2}{1+t_1^3}\right) \\
\frac{3t_0^2}{1+t_0^3} &= \frac{3t_1^2}{1+t_1^3} \\
t_0\left(\frac{3t_0}{1+t_0^3}\right) &= \frac{3t_1^2}{1+t_1^3} \\
t_0\left(\frac{3t_1}{1+t_1^3}\right) &= \frac{3t_1^2}{1+t_1^3} \\
t_0 t_1 &= t_1^2 \\
t_0 &= t_1.
\end{aligned}$$

So we proved that if $\alpha(t_0) = \alpha(t_1)$ then $t_0 = t_1$, which means that α is injective.

Each component of $\alpha(t)$ is a division of polynomials, which would be discontinuous if and only if the denominator $1 + t^3 = 0$ for some t, but this only happens for $t = -1$, which does not belong to the domain of α, so α is continuous.

Let's prove that the inverse α^{-1} is not continuous on $\alpha((-1, \infty))$. α^{-1} is continuous if for any sequence $\{a_n\}_{n\in\mathbb{N}} \subset \alpha((-1, \infty))$, with $a_n \to a \in \alpha((-1, \infty))$ we have $\alpha^{-1}(a_n) \to \alpha^{-1}(a)$.

Choose the sequence $\{a_n\}_{n\in\mathbb{N}}$ defined by $a_n = \alpha(n)$, then

$$\begin{aligned}
\lim_{n\to\infty} a_n &= \lim_{n\to\infty} \alpha(n) \\
&= \lim_{n\to\infty} \left(\frac{3n}{1+n^3}, \frac{3n^2}{1+n^3}\right) \\
&= (0, 0) \\
&= \alpha(0).
\end{aligned}$$

We have then $a_n \to a := \alpha(0) \in \alpha((-1, \infty))$. But $\alpha^{-1}(a_n) = \alpha^{-1}(\alpha(n)) = n \to \infty$, so $\alpha^{-1}(a_n) \nrightarrow \alpha^{-1}(a)$, then α^{-1} is not continuous.

4 Problem 4

Given a circle C and a point P on C, consider the set of all lines Δ_Q such that if $Q \neq P$ is any point on C, the line Δ_Q is the line passing through Q and forming an angle with the normal N_Q at Q equal to the anle of N_Q with PQ (in other words, Δ_Q is obtained by reflecting PQ about the normal N_Q at Q). When $Q = P$ the line Δ_Q is the diameter through P.

(a) Let O be the center of the circle and choose polar coordinates with center O, so that the coordinates of Q are $(\cos\theta, \sin\theta)$. This is illustrated in Figure 4.

We will calculate the coordinates of the second point of intersection M of the line Δ_Q with the circle C.

Let α be the angle between N_Q and PQ, α_1 the angle of inclination of the line PQ, and ϕ the inclination of Δ_Q.

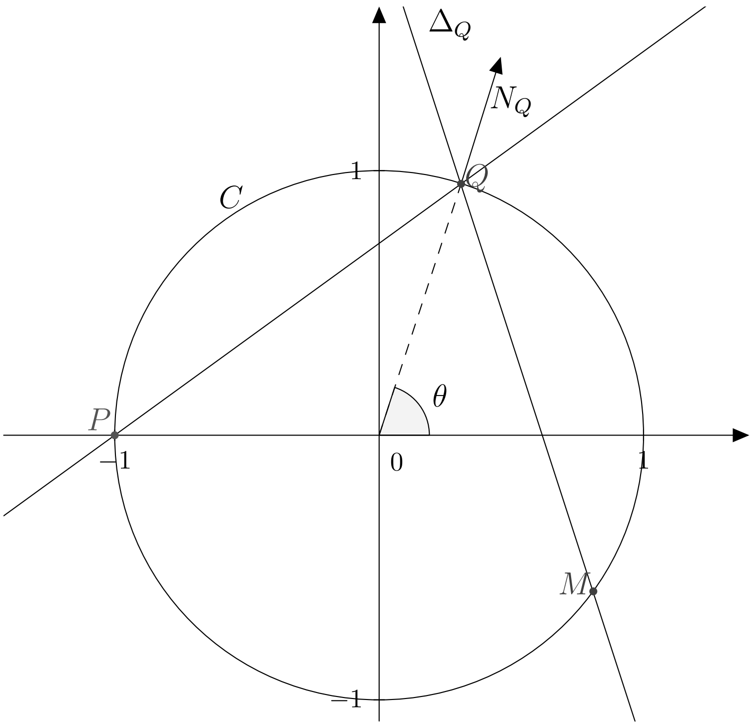

Figure 4: Description of line Δ_Q.

We have

$$\tan \alpha_1 = \frac{\sin\theta}{1+\cos\theta} = \tan(\theta/2),$$

using $\theta \in (-\pi, \pi)$, we have $\alpha_1 = \theta/2$.

The angle of N_Q with the horizontal direction is also θ, so the angle between N_Q and PQ can be obtained by

$$\alpha = \theta - \alpha_1 = \theta - \theta/2 = \theta/2.$$

The angle ϕ of inclination of the line Δ_Q is θ plus the angle between Δ_Q and N_Q, which by definition is equal to the angle α between N_Q and PQ, so

$$\phi = \theta + \alpha = \theta + \theta/2 = 3\theta/2.$$

Then Δ_Q can be defined by

$$\begin{aligned} r(t) &= Q + t(\cos\phi, \sin\phi) \\ &= (\cos\theta, \sin\theta) + t(\cos(3\theta/2), \sin(3\theta/2)) \\ &= (\cos\theta + t\cos(3\theta/2), \sin\theta + t\sin(3\theta/2)). \end{aligned}$$

The points of intersection of Δ_Q and the circle C are those $(x, y) = r(t)$ where $x^2+y^2 =$

1, so

$$\begin{aligned}
x^2+y^2 &= 1\\
(\cos\theta + t\cos(3\theta/2))^2 + (\sin\theta + t\sin(3\theta/2))^2 &= 1\\
\cos^2\theta + 2\cos\theta t\cos(3\theta/2) + t^2\cos^2(3\theta/2) + &\\
\sin^2\theta + 2\sin\theta t\sin(3\theta/2) + t^2\sin^2(3\theta/2) &= 1\\
(\cos^2\theta + \sin^2\theta) + 2t(\cos\theta\cos(3\theta/2) + \sin\theta\sin(3\theta/2)) + t^2(\cos^2(3\theta/2) + \sin^2(3\theta/2)) &= 1\\
1 + 2t(\cos\theta\cos(3\theta/2) + \sin\theta\sin(3\theta/2)) + t^2 &= 1\\
1 + 2t(\cos(\theta - 3\theta/2)) + t^2 &= 1\\
1 + 2t(\cos(-\theta/2)) + t^2 &= 1\\
2t(\cos(-\theta/2)) + t^2 &= 0\\
t(2(\cos(-\theta/2)) + t) &= 0\\
t(2(\cos(\theta/2)) + t) &= 0
\end{aligned}$$

then we have $t = 0$ (when $(x, y) = Q$), or

$$\begin{aligned}
2(\cos(\theta/2)) + t &= 0\\
t &= -2(\cos(\theta/2)),
\end{aligned}$$

so the other point of intersection is

$$\begin{aligned}
M &= (\cos\theta + t\cos(3\theta/2), \sin\theta + t\sin(3\theta/2))\\
&= (\cos\theta - 2(\cos(\theta/2))\cos(3\theta/2), \sin\theta - 2(\cos(\theta/2))\sin(3\theta/2))\\
&= (\cos\theta - 2(1/2)(\cos(\theta/2 - 3\theta/2) + \cos(\theta/2 + 3\theta/2)),\\
&\quad (\sin\theta - 2(1/2)(\sin(3\theta/2 - \theta/2) + \sin(3\theta/2 + \theta/2)))\\
&= (\cos\theta - (\cos(-\theta) + \cos(2\theta)), \sin\theta - (\sin\theta + \sin(2\theta)))\\
&= (\cos\theta - \cos(-\theta) - \cos(2\theta), \sin\theta - \sin\theta - \sin(2\theta))\\
&= (\cos\theta - \cos\theta - \cos(2\theta), -\sin(2\theta))\\
&= (-\cos(2\theta), -\sin(2\theta)).
\end{aligned}$$

(b) Let's write the equation of the line Δ_Q as $y = ax + b$, using the points Q and M to find the values of a and b. Putting each point in the equation we have

$$\begin{cases}\sin\theta = a\cos\theta + b\\ -\sin(2\theta) = a(-\cos(2\theta)) + b\end{cases}$$

Subtracting the equations we obtain:

$$\sin\theta + \sin(2\theta) = a(\cos\theta + \cos(2\theta)).$$

Multiplying the first equation by $\cos(2\theta)$ and the second by $\cos\theta$ we have:

$$\begin{cases}\sin\theta\cos(2\theta) = (a\cos\theta + b)\cos(2\theta)\\ -\sin(2\theta)\cos\theta = (a(-\cos(2\theta)) + b)\cos\theta\end{cases}$$

Summing these equations we have:

$$\begin{aligned}\sin\theta\cos(2\theta) - \sin(2\theta)\cos\theta &= (a\cos\theta + b)\cos(2\theta) + (a(-\cos(2\theta)) + b)\cos\theta \\ \sin(\theta - 2\theta) &= b(\cos(2\theta) + \cos\theta) \\ \sin(-\theta) &= b(\cos(2\theta) + \cos\theta) \\ -\sin(\theta) &= b(\cos(2\theta) + \cos\theta).\end{aligned}$$

Now, multiplying $y = ax + b$ by $(\cos\theta + \cos 2\theta)$:

$$\begin{aligned}y(\cos\theta + \cos 2\theta) &= (ax + b)(\cos\theta + \cos 2\theta) \\ y(\cos\theta + \cos 2\theta) &= a(\cos\theta + \cos 2\theta)x + b(\cos\theta + \cos 2\theta) \\ y(\cos\theta + \cos 2\theta) &= (\sin\theta + \sin 2\theta)x - \sin\theta,\end{aligned}$$

then the equation of Δ_Q can be written as

$$(\sin\theta + \sin 2\theta)x - (\cos\theta + \cos 2\theta)y - \sin\theta = 0.$$

Given $\theta \neq 0$, the intersection of Δ_Q and the x-axis is the point $(x, y = 0)$, where, using the equation of Δ_Q, we have:

$$\begin{aligned}(\sin\theta + \sin 2\theta)x - (\cos\theta + \cos 2\theta)0 - \sin\theta &= 0 \\ (\sin\theta + \sin 2\theta)x - \sin\theta &= 0 \\ x &= \frac{\sin\theta}{(\sin\theta + \sin 2\theta)}.\end{aligned}$$

When $\theta \to 0$ we can calculate the limit x_0 of the x coordinate of this intersection:

$$\begin{aligned}x_0 &= \lim_{\theta\to 0} \frac{\sin\theta}{(\sin\theta + \sin 2\theta)} \\ &= \lim_{\theta\to 0} \frac{\cos\theta}{(\cos\theta + 2\cos 2\theta)} \qquad \text{(using l'Hopital's rule)} \\ &= 1/3.\end{aligned}$$

Then, when θ approximates to 0, the intersection point tends to $(1/3, 0)$.

(c) A line is given by an equation

$$ux + vy + w = 0,$$

where u, v, w are differentiable functions of a parameter θ. If this line is tangent to a curve Γ at some point $(\bar{x}, \bar{y})$, then (x,y) will be the limit of the intersection of the line of equation

$$u(\theta)x + v(\theta)y + w(\theta) = 0$$

with the line of equation

$$u(\theta + \epsilon)x + v(\theta + \epsilon)y + w(\theta + \epsilon) = 0$$

when $\epsilon \neq 0$ tends to zero.

Given $\epsilon \neq 0$, denote (x_ϵ, y_ϵ) the intersection of these lines. We have then $\lim_{\epsilon\to 0}(x_\epsilon, y_\epsilon) = (\bar{x}, \bar{y})$.

We will prove that $(\bar{x}, \bar{y})$ is the solution of the two equations

$$u(\theta)x + v(\theta)y + w(\theta) = 0 \tag{1}$$

$$u'(\theta)x + v'(\theta)y + w'(\theta) = 0 \tag{2}$$

The first equation is trivially satisfied for $(\bar{x}, \bar{y})$, because this point belongs to the curve defined by that equation.

We know that every (x_ϵ, y_ϵ) satisfies

$$u(\theta + \epsilon)x + v(\theta + \epsilon)y + w(\theta + \epsilon) = 0.$$

Let's use Taylor expansion for $u(\theta + \epsilon), v(\theta + \epsilon)$ and $w(\theta + \epsilon)$:

$$\begin{aligned} u(\theta + \epsilon) &= u(\theta) + \epsilon u'(\theta) + r_u(\epsilon) \\ v(\theta + \epsilon) &= v(\theta) + \epsilon v'(\theta) + r_v(\epsilon) \\ w(\theta + \epsilon) &= w(\theta) + \epsilon w'(\theta) + r_w(\epsilon), \end{aligned}$$

where

$$\begin{aligned} \lim_{\epsilon\to 0} \frac{r_u(\epsilon)}{\epsilon} &= 0 \\ \lim_{\epsilon\to 0} \frac{r_v(\epsilon)}{\epsilon} &= 0 \\ \lim_{\epsilon\to 0} \frac{r_w(\epsilon)}{\epsilon} &= 0. \end{aligned}$$

Using those expansions, we get

$$\begin{aligned} (u(\theta) + \epsilon u'(\theta) + r_u(\epsilon))x_\epsilon + \\ (v(\theta) + \epsilon v'(\theta) + r_v(\epsilon))y_\epsilon + \\ w(\theta) + \epsilon w'(\theta) + r_w(\epsilon) &= 0 \\ u(\theta)x_\epsilon + \epsilon u'(\theta)x_\epsilon + r_u(\epsilon)x_\epsilon + \\ v(\theta)y_\epsilon + \epsilon v'(\theta)y_\epsilon + r_v(\epsilon)y_\epsilon + \\ w(\theta) + \epsilon w'(\theta) + r_w(\epsilon) &= 0 \\ (u(\theta)x_\epsilon + v(\theta)y_\epsilon + w(\theta)) + \\ \epsilon(u'(\theta)x_\epsilon + v'(\theta)y_\epsilon + w'(\theta)) + \\ r_u(\epsilon)x_\epsilon + r_v(\epsilon)y_\epsilon + r_w(\epsilon) &= 0. \end{aligned}$$

As the point (x_ϵ, y_ϵ) also belongs to the line defined by $u(\theta)x + v(\theta)y + w(\theta) = 0$, we get

$$\begin{aligned} \epsilon(u'(\theta)x_\epsilon + v'(\theta)y_\epsilon + w'(\theta)) + r_u(\epsilon)x_\epsilon + r_v(\epsilon)y_\epsilon + r_w(\epsilon) &= 0 \\ (u'(\theta)x_\epsilon + v'(\theta)y_\epsilon + w'(\theta)) + \frac{r_u(\epsilon)x_\epsilon}{\epsilon} + \frac{r_v(\epsilon)y_\epsilon}{\epsilon} + \frac{r_w(\epsilon)}{\epsilon} &= 0. \end{aligned}$$

Applying the limit as ϵ goes to zero:

$$\begin{aligned} \lim_{\epsilon\to 0}\left((u'(\theta)x_\epsilon + v'(\theta)y_\epsilon + w'(\theta)) + \frac{r_u(\epsilon)x_\epsilon}{\epsilon} + \frac{r_v(\epsilon)y_\epsilon}{\epsilon} + \frac{r_w(\epsilon)}{\epsilon}\right) &= 0 \\ u'(\theta)\bar{x} + v'(\theta)\bar{y} + w'(\theta) &= 0. \end{aligned}$$

So $(\bar{x}, \bar{y})$ also satisfies equation 2.

If we define $F(\theta, x, y) := (\sin 2\theta + \sin\theta)x - (\cos 2\theta + \cos\theta)y - \sin\theta$, we can see that a line Δ_Q, with $Q = (\cos\theta, \sin\theta)$, can be expressed as $F(\theta, x, y) = 0$.

Given a value of θ, we can find the point $(\bar{x}, \bar{y})$ that is the limit of the intersection of the lines $F(\theta, x, y) = 0$ and $F(\theta + \epsilon, x, y) = 0$. The envelope of the lines Δ_Q are all the points that satisfy this relation for some θ.

Let's define

$$\begin{aligned} u(\theta) &= \sin 2\theta + \sin\theta \\ v(\theta) &= -\cos 2\theta - \cos\theta \\ w(\theta) &= -\sin\theta, \end{aligned}$$

so we can write $F(\theta, x, y) = u(\theta)x + v(\theta)y + w(\theta)$.

Using the result above, we see that the points in the envelope are solutions of the equations

$$\begin{aligned} F(\theta, x, y) &= u(\theta)x + v(\theta)y + w(\theta) = 0 \\ \frac{\partial F}{\partial \theta}(\theta, x, y) &= u'(\theta)x + v'(\theta)y + w'(\theta) = 0 \end{aligned}$$

for each θ. We will omit the parameter θ to simplify the notation. These equations can be written as

$$\begin{pmatrix} u & v \\ u' & v' \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -w \\ -w' \end{pmatrix}.$$

Solving this system we get:

$$x = \frac{\det\begin{pmatrix} -w & v \\ -w' & v' \end{pmatrix}}{\det\begin{pmatrix} u & v \\ u' & v' \end{pmatrix}} \qquad y = \frac{\det\begin{pmatrix} u & -w \\ u' & -w' \end{pmatrix}}{\det\begin{pmatrix} u & v \\ u' & v' \end{pmatrix}}$$

$$x = \frac{-v'w + vw'}{uv' - u'v} \qquad y = \frac{-uw' + u'w}{uv' - u'v}$$

where

$$\begin{aligned} uv' - u'v &= (\sin 2\theta + \sin\theta)(2\sin 2\theta + \sin\theta) - (2\cos 2\theta + \cos\theta)(-\cos 2\theta - \cos\theta) \\ &= 2\sin^2 2\theta + 3\sin\theta\sin 2\theta + \sin^2\theta + 2\cos^2 2\theta + 3\cos\theta\cos 2\theta + \cos^2\theta \\ &= 2(\sin^2 2\theta + \cos^2 2\theta) + (\sin^2\theta + \cos^2\theta) + 3(\sin\theta\sin 2\theta + \cos\theta\cos 2\theta) \\ &= 2 + 1 + 3\cos(2\theta - \theta) \\ &= 3(1 + \cos\theta). \end{aligned}$$

$$\begin{aligned} -v'w + vw' &= -(2\sin 2\theta + \sin\theta)(-\sin\theta) + (-\cos 2\theta - \cos\theta)(-\cos\theta) \\ &= 2\sin\theta\sin 2\theta + \sin^2\theta + \cos\theta\cos 2\theta + \cos^2\theta \\ &= (2\sin\theta\sin 2\theta) + (\cos\theta\cos 2\theta) + (\sin^2\theta + \cos^2\theta) \\ &= \cos(2\theta - \theta) - \cos(2\theta + \theta) + (1/2)(\cos(2\theta - \theta) + \cos(2\theta + \theta)) + 1 \\ &= \cos(\theta) - \cos(3\theta) + (1/2)(\cos(\theta) + \cos(3\theta)) + 1 \\ &= (3/2)\cos\theta - (1/2)\cos 3\theta + 1 \end{aligned}$$

Using some trigonometric relations we can do

$$\begin{aligned}\cos 3\theta &= \cos(2\theta + \theta)\\ &= \cos 2\theta \cos\theta - \sin 2\theta \sin\theta\\ &= (2\cos^2 - 1)\cos\theta - 2\sin\theta\cos\theta\sin\theta\\ &= 2\cos^3\theta - \cos\theta - 2\sin^2\theta\cos\theta\\ &= 2\cos^3\theta - \cos\theta - 2(1-\cos^2\theta)\cos\theta\\ &= 2\cos^3\theta - \cos\theta - 2\cos\theta + 2\cos^3\theta\\ &= 4\cos^3\theta - 3\cos\theta,\end{aligned}$$

then we can continue

$$\begin{aligned}-v'w + vw' &= (3/2)\cos\theta - (1/2)(4\cos^3\theta - 3\cos\theta) + 1\\ &= 3\cos\theta + 2\cos^3\theta + 1\\ &= \cos\theta + 2\cos\theta - 2\cos\theta\cos^2\theta + 1\\ &= (1+\cos\theta) + 2\cos\theta(1-\cos^2\theta)\\ &= (1+\cos\theta) + 2\cos\theta(1+\cos\theta)(1-\cos\theta)\\ &= (1+\cos\theta)(1+2\cos\theta(1-\cos\theta)).\end{aligned}$$

So we get

$$\begin{aligned}x &= \frac{-v'w + vw'}{uv' - u'v}\\ &= \frac{(1+\cos\theta)(1+2\cos\theta(1-\cos\theta))}{3(1+\cos\theta)}\\ &= 1/3 + 2/3\cos\theta(1-\cos\theta).\end{aligned}$$

$$\begin{aligned}-uw' + u'w &= -(\sin 2\theta + \sin\theta)(-\cos\theta) + (2\cos 2\theta + \cos\theta)(-\sin\theta)\\ &= \sin 2\theta\cos\theta + \sin\theta\cos\theta - 2\sin\theta\cos 2\theta - \sin\theta\cos\theta\\ &= \sin 2\theta\cos\theta - 2\sin\theta\cos 2\theta\\ &= (1/2)(\sin(2\theta-\theta) + \sin(2\theta+\theta)) - (\sin(\theta - 2\theta) + \sin(\theta + 2\theta))\\ &= (1/2)(\sin\theta + \sin 3\theta) - \sin(-\theta) - \sin 3\theta\\ &= (3/2)\sin\theta - (1/2)\sin 3\theta.\end{aligned}$$

We can also calculate

$$\begin{aligned}\sin 3\theta &= \sin(2\theta + \theta)\\ &= \sin 2\theta\cos\theta + \sin\theta\cos 2\theta\\ &= 2\sin\theta\cos^2\theta + \sin\theta(2\cos^2\theta - 1)\\ &= 4\sin\theta\cos^2\theta - \sin\theta\\ &= \sin\theta(4\cos^2\theta - 1).\end{aligned}$$

So we have

$$\begin{aligned} -uw' + u'w &= (3/2)\sin\theta - (1/2)(\sin\theta(4\cos^2\theta - 1)) \\ &= (1/2)\sin\theta(3 - (4\cos^2\theta - 1)) \\ &= (1/2)\sin\theta(4 - 4\cos^2\theta) \\ &= 2\sin\theta(1 - \cos^2\theta) \\ &= 2\sin\theta(1 + \cos\theta)(1 - \cos\theta) \end{aligned}$$

Then we get y

$$\begin{aligned} y &= \frac{-uw' + u'w}{uv' - u'v} \\ &= \frac{2\sin\theta(1 + \cos\theta)(1 - \cos\theta)}{3(1 + \cos\theta)} \\ &= (2/3)\sin\theta(1 - \cos\theta). \end{aligned}$$

So the envelope of the lines Δ_Q can be defined by the parametric equations

$$\begin{aligned} x &= 1/3 + 2/3\cos\theta(1 - \cos\theta) \\ y &= (2/3)\sin\theta(1 - \cos\theta). \end{aligned}$$

To see that this is a cardioid, we can make a simple changing of variables:

$$\begin{aligned} x' &= -(3/2)(x - 1/3), \\ y' &= (3/2)y, \\ \theta' &= \pi - \theta, \end{aligned}$$

such that

$$\begin{aligned} x' &= \cos(\theta')(1 + \cos(\theta')) \\ y' &= \sin(\theta')(1 + \cos(\theta')), \end{aligned}$$

these equations define a cardioid translated, scaled and with a shift in θ, its cusp is $0 = x' = (-3/2)(x - 1/3)$, $0 = y' = (3/2)y$, i.e. $x = 1/3$, $y = 0$.